Higgs mechanism for new massive gravity and Weyl-invariant extensions of higher-derivative theories

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September 26, 201, Georgia
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2. Brief Review of Weyl Transformation
   - Basics of Weyl-invariance

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   - Weyl-invariant Einstein-Gauss-Bonnet Theory and Born-Infeld NMG

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Subtle problem: mass of graviton.

Supplying a Pauli-Fierz type hard mass to graviton, $m^2(h_{\mu\nu}^2 - h^2)$:

- General covariance is lost.
- Ghosts appear beyond the three level.
- $m^2 = 0$ is discretely disconnected from $m^2 \to 0$.

Solution: non-Fierz-Pauli masses for the gravitons.
In 2009, E. Bergshoeff *et al* presented, New Massive gravity [NMG] which is defined in three dimensional spacetime.

- It describes the non-linear, generally covariant extension of the Pauli-Fierz massive gravity

\[
I_{NMG} = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left[ \sigma R - 2\lambda m^2 + \frac{1}{m^2} \left( R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \right]. \tag{1}
\]

- It gives, at linearized level, a massive graviton with 2 degrees of freedom both around its flat and (anti)-de Sitter vacua.


- In AdS, it is only unitary either in the bulk or on the boundary. (Bergshoeff, Hohm and Townsend, PRD, 2010)

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- In AdS, it is only unitary either in the bulk or on the boundary. (Bergshoeff, Hohm and Townsend, PRD, 2010)
Is there any Higgs-type mechanism for the spin 2 particles?
In this paper, we present such a Higgs-type mechanism for massive spin-2 particle by:

- First, constructing the Weyl-invariant NMG and then showing that the vacua of the theory breaks Weyl-symmetry.
- Symmetry breaking depends on the backgrounds (assuming one does not add any explicit symmetry breaking term by hand).
- The unitary analysis should be done in detail.
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Higgs mechanism for new massive gravity and Weyl-invariant extensions
As a sample, in order to make the kinetic part of the scalar field action

\[ S_\Phi = -\frac{1}{2} \int d^n x \sqrt{-g} \partial_\mu \Phi \partial_\nu \Phi g^{\mu\nu}. \]  \hspace{1cm} (2)

To make the action Weyl-invariant, one should have invariance under

\[ g_{\mu\nu} \rightarrow g'_{\mu\nu} = e^{2\lambda(x)} g_{\mu\nu}, \quad \Phi \rightarrow \Phi' = e^{-\frac{(n-2)}{2} \lambda(x)} \Phi, \]  \hspace{1cm} (3)

the derivatives should be replaced with the (real) gauge covariant derivatives

\[ \mathcal{D}_\mu \Phi = \partial_\mu \Phi - \frac{n-2}{2} A_\mu \Phi, \quad \mathcal{D}_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} + 2 A_\mu g_{\alpha\beta}, \]  \hspace{1cm} (4)

where \( A_\mu \) is the Weyl’s gauge field which transforms as

\[ A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \lambda(x). \]  \hspace{1cm} (5)
Then, conformal transformation of the gauge covariant derivatives become

\[(\mathcal{D}_\mu g_{\alpha\beta})' = e^{2\lambda(x)}\mathcal{D}_\mu g_{\alpha\beta}, \quad (\mathcal{D}_\mu \Phi)' = e^{-\frac{(n-2)}{2}\lambda(x)}\mathcal{D}_\mu \Phi. \quad (6)\]

> Since the field strength \(F_{\mu\nu}\) is gauge invariant, the Maxwell-type action needs a compensating Weyl scalar

\[S_{A\mu} = -\frac{1}{2} \int d^n x \sqrt{-g} \Phi \frac{2(n-4)}{n-2} F_{\mu\nu}^2, \quad (7)\]

where \(F_{\mu\nu}^2 = F_{\mu\nu} F^{\mu\nu}\).
In order to implement the Weyl invariance into gravity, one should define a Weyl-invariant connection with the help of regular Christoffel connection and gauge field:

\[ \tilde{\Gamma}_\mu^{\lambda \nu} = \frac{1}{2} g^{\lambda \sigma} \left( \mathcal{D}_\mu g_{\sigma \nu} + \mathcal{D}_\nu g_{\mu \sigma} - \mathcal{D}_\sigma g_{\mu \nu} \right). \]  

Then, the Weyl-invariant “Riemann tensor” becomes

\[ \tilde{R}^\mu_{\nu \rho \sigma}[g, A] = \partial_\rho \tilde{\Gamma}^\mu_{\nu \sigma} - \partial_\sigma \tilde{\Gamma}^\mu_{\nu \rho} + \tilde{\Gamma}^\mu_{\lambda \rho} \tilde{\Gamma}^{\lambda}_{\nu \sigma} - \tilde{\Gamma}^\mu_{\lambda \sigma} \tilde{\Gamma}^{\lambda}_{\nu \rho} = R^\mu_{\nu \rho \sigma} + \delta^\mu_{\nu} F_{\rho \sigma} + 2\delta^\mu_{\nu} [\sigma \nabla_\rho] A_\sigma + 2g_{\nu[\rho} \nabla_{\sigma]} A^\mu_{\rho]}
\begin{align*}
&+ 2A_{[\sigma} \delta_{\rho]}^\mu A_{\nu]} + 2g_{\nu[\sigma} A_{\rho]} A^\mu + 2g_{\nu[\rho} \delta_{\sigma]} A^2 \end{align*} \]
Similarly, the Weyl-invariant “Ricci-tensor” becomes

\[ \tilde{R}_{\nu\sigma}[g, A] = \tilde{R}^{\mu}_{\nu\mu\sigma}[g, A] \]

\[ = R_{\nu\sigma} + F_{\nu\sigma} - (n - 2) \left[ \nabla_\sigma A_\nu - A_\nu A_\sigma + A^2 g_{\nu\sigma} \right] \]

\[ - g_{\nu\sigma} \nabla \cdot A. \]  \hspace{1cm} (10)

Finally, “the Ricci-scalar” is computed as

\[ \tilde{R}[g, A] = R - 2(n - 1) \nabla \cdot A - (n - 1)(n - 2)A^2, \] \hspace{1cm} (11)

which is not Weyl invariant, but transforms as

\[ (\tilde{R}[g, A])' = e^{-2\lambda} \tilde{R}[g, A]. \]
Since, the curvature tensor is not Weyl-invariant, the Weyl-invariant Einstein-Hilbert action requires a compensating Weyl scalar:

\[
S = \int d^n x \sqrt{-g} \Phi^2 \tilde{\mathcal{R}}[g, A]
= \int d^n x \sqrt{-g} \Phi^2 \left( R - 2(n - 1) \nabla \cdot A - (n - 1)(n - 2)A^2 \right).
\]

(12)

After eliminating the gauge field by using its field equation

\[
A_\mu = \frac{2}{n - 2} \partial_\mu \ln \Phi.
\]

(13)
one gets “the conformally coupled scalar-tensor” theory

\[ S = \int d^n x \sqrt{-g} \left( \Phi^2 R + 4 \frac{(n-1)}{n-2} \partial_\mu \Phi \partial^\mu \Phi \right). \]  \hspace{1cm} (14)

For particular constant value of the scalar field, the action (14) reduces to general relativity without cosmological constant. For non-zero cosmological constant case, a Weyl-invariant potential should be added to the scalar field action:

\[ S_\Phi = -\frac{1}{2} \int d^n x \sqrt{-g} \left( D_\mu \Phi D^{\mu} \Phi + \nu \Phi \frac{2n}{n-2} \right), \]  \hspace{1cm} (15)

where \( \nu \geq 0 \) is a dimensionless coupling constant.
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Higgs mechanism for new massive gravity and Weyl-invariant extensions
Using the tools developed in the previous section, the Weyl-invariant extension of NMG can be written as

\[ \tilde{S}_{NMG} = \int d^3x \sqrt{-g} \left[ \sigma \Phi^2 \tilde{R} + \Phi^{-2} \left( \tilde{R}_{\mu\nu}^2 - \frac{3}{8} \tilde{R}^2 \right) \right] + S_\Phi, \quad (16) \]

where the last term is the scalar action given in (15).
The explicit form of Weyl-invariant NMG

\[ \tilde{S}_{NMG} = \int d^3 x \sqrt{-g} \left\{ \sigma \Phi^2 \left( R - 4 \nabla \cdot A - 2A^2 \right) \right. \\
+ \Phi^{-2} \left[ R_{\mu \nu}^2 - \frac{3}{8} R^2 - 2R^{\mu \nu} \nabla_{\mu} A_{\nu} + 2R^{\mu \nu} A_{\mu} A_{\nu} \right. \right. \\
+ R \nabla \cdot A - \frac{1}{2} RA^2 + 2F_{\mu \nu}^2 + (\nabla_{\mu} A_{\nu})^2 \right. \right. \\
- 2A_{\mu} A_{\nu} \nabla^{\mu} A^{\nu} - (\nabla \cdot A)^2 + \frac{1}{2} A^4 \right\} + S_{\Phi}. \tag{17} \]

For \( A_{\mu} = 0, \Phi = \sqrt{m} \) and \( \nu = 2\lambda \Rightarrow \text{NMG with } \kappa = m^{-1/2}. \)
As we see, the Newton’s constant is related to the mass of graviton. So does this limit arise from the vacuum solution of the theory? (in another word, without adding an explicit Weyl symmetry breaking term, does the vacuum break the Weyl symmetry?) ⇒ Therefore, to see this, We need field equations:
Variation with respect to $g^{\mu \nu}$ gives

\[
\sigma \Phi^2 G_{\mu \nu} + \sigma g_{\mu \nu} \Box \Phi^2 - \sigma \nabla_\mu \nabla_\nu \Phi^2 - 4\sigma \Phi^2 \nabla_\mu A_\nu \\
+ 2\sigma g_{\mu \nu} \Phi^2 \nabla \cdot A - 2\sigma \Phi^2 A_\mu A_\nu + \sigma g_{\mu \nu} \Phi^2 A^2 \\
+ 2\Phi^{-2} [R_{\mu \sigma \nu \alpha} - \frac{1}{4} g_{\mu \nu} R_{\sigma \alpha}] R^{\sigma \alpha} \\
+ \Box(\Phi^{-2} G_{\mu \nu}) + \frac{1}{4} [g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu] \Phi^{-2} R \\
+ g_{\mu \nu} G^{\sigma \alpha} \nabla_\sigma \nabla_\alpha \Phi^{-2} - 2 G^{\sigma \nu} \nabla_\sigma \nabla_\mu \Phi^{-2} \\
- 2(\nabla_\mu G^{\sigma \nu})(\nabla_\sigma \Phi^{-2}) + \frac{3}{16} g_{\mu \nu} \Phi^{-2} R^2 \\
+ \ldots = - \frac{1}{\sqrt{-g}} \frac{\delta S_\Phi}{\delta g^{\mu \nu}}.
\]
Variation with respect to $\Phi$ yields

$$2\sigma \Phi \left( R - 4 \nabla \cdot A - 2A^2 \right) - 2\Phi^{-3} \left[ R^2_{\mu\nu} - \frac{3}{8} R^2 \right. $$

$$- 2R^\mu\nu \nabla_\mu A_\nu + 2R^\mu\nu A_\mu A_\nu + R \nabla \cdot A - \frac{1}{2} RA^2 $$

$$+ 2F^2_{\mu\nu} + (\nabla_\mu A_\nu)^2 - 2A_\mu A_\nu \nabla^\mu A^\nu - (\nabla \cdot A)^2 $$

$$+ \frac{1}{2} A^4 \right] = - \frac{1}{\sqrt{-g}} \frac{\delta S_\Phi}{\delta \Phi}.$$ (19)
And finally, the Weyl gauge-field equation becomes

\[
\begin{align*}
-4\nabla_{\mu} \Phi^2 + 4\Phi^2 A_{\mu} + 2R^\nu_{\mu} \nabla_\nu \Phi^{-2} + 4R_{\mu\nu} A^\nu \\
- R\nabla_{\mu} \Phi^{-2} - \Phi^{-2} RA_{\mu} + 8\nabla^\nu (\Phi^{-2} \nabla_\mu A_\nu) \\
- 10\nabla^\nu (\Phi^{-2} \nabla_\nu A_\mu) + 2\nabla_\alpha (\Phi^{-2} A^\alpha A_\mu) \\
- 2\Phi^{-2} (\nabla_\mu A_\nu) A^\nu - 2\Phi^{-2} (\nabla_\nu A_\mu) A^\nu \\
+ 2\nabla_{\mu} (\Phi^{-2} \nabla \cdot A) + 2A_\mu A^2 = -\frac{1}{\sqrt{-g}} \frac{\delta S_\Phi}{\delta A_\mu}.
\end{align*}
\] (20)
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In order to avoid breaking local Lorentz invariance of the vacuum, let us set $F_{\mu\nu} = 0$ and choose the gauge for which $A_\mu = 0$.

For the case of (Anti)-de Sitter background, let us set

$$\Phi \equiv \sqrt{m}, \quad R_{\mu\nu} = 2\Lambda g_{\mu\nu}.$$ (21)

The gauge field equation is automatically satisfied and the other two field equations give the same equation

$$\nu m^4 - 4\sigma m^2 \Lambda - \Lambda^2 = 0.$$ (22)
1st case: Suppose that $\Lambda$ is known. Then, one gets

$$m^2_{\pm} = \frac{2\sigma \Lambda}{\nu} \pm \frac{|\Lambda|}{\nu} \sqrt{4\sigma^2 + \nu}. \quad (23)$$

Since, $\kappa = m^{-1/2} > 0$, the negative root is not allowed for both sign of $\Lambda$. And also, taken from the previous works, the mass of the graviton is fixed as

$$M_g^2 = -\sigma m^2_+ + \frac{\Lambda}{2}. \quad (24)$$

For the unitarity of the theory in dS the Higuchi bound $M_g^2 \geq \Lambda > 0$ must be satisfied and for unitarity in AdS Breitenlohner-Freedman bound $M_g^2 \geq \Lambda$ must be satisfied. Since these two forms are formally equal, one has

$$-4\text{sign}(\Lambda) - 2\sigma \sqrt{4 + \nu} \geq \text{sign}(\Lambda)\nu. \quad (25)$$

For $\Lambda > 0$, one must have $\sigma = -1$. For $\Lambda < 0$, both signs of $\sigma$ are allowed.
• **2nd Case:** Suppose that the vacuum expectation value of the scalar field is known. Then the cosmological constant is evaluated as

\[ \Lambda_{\pm} = m^2 \left[ -2\sigma \pm \sqrt{4 + \nu} \right]. \]

(26)

Moreover, for the case \( \nu = 0 \rightarrow \Lambda = -4\sigma m^2 \):

- in de-Sitter, \( \sigma = -1 \) is allowed; \( M_g^2 \) becomes \(-3\sigma m^2\) but *Higuchi bound* is not satisfied \( \rightarrow \) the theory is *not unitary*.
- in Anti-de Sitter, \( \sigma = +1 \) is allowed and Breitenlohner-Freedman bound is satisfied \( \rightarrow \) the theory is *unitary*. 
For the case of Flat background:

- Around the flat background, $\Lambda = 0 \rightarrow m = 0$; the Weyl symmetry of the Lagrangian is not broken by the vacuum solution. One way to break symmetry is to add an explicit mass term to the Lagrangian.
- Alternatively, one can check whether the radiative corrections do break the symmetry as done by Coleman and Weinberg in the massless $\Phi^4$ theory in four dimensions. However, we are working in three dimensions. Fortunately, the three dimensional computation for $\nu \Phi^6$ theory was carried out by P.N. Tan, B. Tekin and Y. Hosotani (1996 and 1997).
After the calculations at two-loop level, they found the effective scalar potential as

\[ V_{\text{eff}} = \nu(\mu)\Phi^6 + \frac{7\hbar^2}{120\pi^2}\nu(\mu)^2\Phi^6 \left( \ln \frac{\Phi^4}{\mu^2} - \frac{49}{5} \right). \]  

(27)

Thus, it is obvious that the minimum of the potential is away from \( \Phi = 0 \) and the Weyl symmetry is broken as we desire.
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With the help of specified-order of compensating scalar field, the generic Weyl-invariant quadratic gravity action is obtained as

$$\tilde{S}_{\text{quadratic}} = \int d^n x \sqrt{-g} \Phi \frac{2(n-4)}{n-2} \left[ \alpha \tilde{R}^2 + \beta \tilde{R}_{\mu\nu}^2 + \gamma \tilde{R}_{\mu\nu\rho\sigma}^2 \right],$$

(28)

where the explicit form of the curvature square terms read as

$$\tilde{R}^2 = R^2 - 4(n-1)R(\nabla \cdot A) - 2(n-1)(n-2)RA^2$$

$$+ 4(n-1)^2(\nabla \cdot A)^2 + 4(n-1)^2(n-2)A^2(\nabla \cdot A)$$

$$+ (n-1)^2(n-2)^2A^4,$$

(29)

where $\nabla \cdot A = \nabla_\mu A^\mu$, $A^2 = A_\mu A^\mu$ and $A^4 = A_\mu A^\mu A_\nu A^\nu$. 
\[ \tilde{R}^2_{\mu\nu} = R^2_{\mu\nu} - 2(n - 2)R^{\mu\nu} \nabla_\nu A_\mu - 2R(\nabla \cdot A) + 2(n - 2)R^{\mu\nu} A_\mu A_\nu \\
- 2(n - 2)RA^2 + F^2_{\mu\nu} - 2(n - 2)F^{\mu\nu} \nabla_\nu A_\mu \\
+ (n - 2)^2(\nabla_\nu A_\mu)^2 + (3n - 4)(\nabla \cdot A)^2 \\
- 2(n - 2)^2A_\mu A_\nu \nabla^\mu A^\nu + (4n - 6)(n - 2)A^2(\nabla \cdot A) \\
+ (n - 2)^2(n - 1)A^4. \]

(30)
\[ \tilde{R}^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} - 8 R^{\mu\nu} \nabla_\mu A_\nu + 8 R^{\mu\nu} A_\mu A_\nu - 4 R A^2 + n F_{\mu\nu}^2 \\
+ 4(n - 2)(\nabla_\mu A_\nu)^2 + 4(\nabla \cdot A)^2 + 8(n - 2) A^2 (\nabla \cdot A) \\
- 8(n - 2) A_\mu A_\nu \nabla^\mu A^\nu + 2(n - 1)(n - 2) A^4. \] (31)

By using these curvature square terms, one can study any Weyl-invariant quadratic theory. Particularly, n-dimensional the Weyl-invariant Gauss-Bonnet combination can be easily written which for the specific case of n=3 reduces to Maxwell theory

\[ \tilde{R}^2_{\mu\nu\rho\sigma} - 4 \tilde{R}^2_{\mu\nu} + \tilde{R}^2 = -5 F_{\mu\nu}^2. \] (32)
Finally, the Weyl-invariant form of the Born-Infeld NMG is obtained as

\[ S_{BINMG} = -4 \int d^3 x \left[ \sqrt{-\det(\Phi^4 g + \sigma \tilde{G})} - (1 - \frac{\lambda}{2}) \sqrt{-\Phi^4 g} \right], \tag{33} \]

where \( \tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} \) is the Weyl-invariant Einstein tensor.
Expansion of the determinant in terms of the traces yields

\[
\sqrt{-\det(\Phi^4 g + \sigma \tilde{G})} = \sqrt{-\det(\Phi^4 g)} \left(1 - \frac{1}{2} \Phi^{-4} \tilde{R}^{\mu\nu} \left[ - g_{\mu\nu} + \Phi^{-4} \left( \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} \right) + \frac{2}{3} \Phi^{-8} \left( \tilde{R}_{\mu\rho} \tilde{R}^{\rho\nu} - \frac{3}{4} \tilde{R} \tilde{R}_{\mu\nu} + \frac{1}{8} g_{\mu\nu} \tilde{R}^2 \right) \right] \right)^{1/2}
\]

which is exact up to this point. From this expression, one can construct Weyl-invariant theories at any order in the curvature by doing a Taylor series expansion in the curvature.
The Weyl-invariant extension of the generic quadratic theories and Born-Infeld form of New massive gravity (BINMG) are given.

Particularly, for the New Massive Gravity, the Weyl symmetry is spontaneously broken in the (Anti)-de Sitter backgrounds, but in flat backgrounds radiative corrections, at two-loop level, break the Weyl symmetry. *The unitarity analysis should be done in detail.*