Aspects of Liouville field theory with defects

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based on paper

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On classical and semiclassical properties of the Liouville theory with defects.
Topological defects

Defects in two-dimensional conformal field theories can be realized as oriented lines, separating different theories. We are interested in the special class of defects, for which the energy-momentum tensor is continuous across the defect. Denoting the left- and right- moving energy-momentum tensors of the two theories by $T^{(1)}$, $T^{(2)}$, and $\bar{T}^{(1)}$, $\bar{T}^{(2)}$, this condition takes the form:

$$ T^{(1)} = T^{(2)}, \quad \bar{T}^{(1)} = \bar{T}^{(2)}. \quad (1) $$

Inserting a defect in the path integral is equivalent in the operator language to the insertion of an operator $D$ which maps the Hilbert space of CFT 1 to that of CFT 2. Condition (1) can be
considered as implying that the corresponding operator $D$ commutes with the Virasoro modes:

$$DL_m^{(1)} = L_m^{(2)} D \quad \text{and} \quad D\bar{L}_m^{(1)} = \bar{L}_m^{(2)} D.$$ (2)
Classical Liouville theory with defects

Review of Liouville solution

Let us recall some facts on classical Liouville theory.

The action of the Liouville theory is

$$ S = \frac{1}{2\pi i} \int \left( \partial \phi \bar{\partial} \phi + \mu \pi e^{2b\phi} \right) d^2 z. \quad (3) $$

Here we use a complex coordinate $z = \tau + i\sigma$, and $d^2 z \equiv dz \wedge d\bar{z}$ is the volume form.

The field $\phi(z, \bar{z})$ satisfies the classical Liouville equation of motion

$$ \partial \bar{\partial} \phi = \pi \mu e^{2b\phi}. \quad (4) $$
The general solution to (4), also derived below, was given by Liouville in terms of two arbitrary functions \( A(z) \) and \( B(\bar{z}) \)

\[
\phi = \frac{1}{2b} \log \left( \frac{1}{\pi \mu b^2} \frac{\partial A(z) \partial B(\bar{z})}{(A(z) + B(\bar{z}))^2} \right). \tag{5}
\]

The solution (5) is invariant if one transforms \( A \) and \( B \) simultaneously by the following constant Möbius transformations:

\[
A \rightarrow \frac{\zeta A + \beta}{\gamma A + \delta}, \quad B \rightarrow \frac{\zeta B - \beta}{-\gamma B + \delta}, \quad \zeta \delta - \beta \gamma = 1. \tag{6}
\]

Classical expressions for left and right components of the energy-momentum tensor are

\[
T = - (\partial \phi)^2 + b^{-1} \partial^2 \phi, \quad \tag{7}
\]

\[
\bar{T} = - (\bar{\partial} \phi)^2 + b^{-1} \bar{\partial}^2 \phi. \quad \tag{8}
\]

Substituting (5) in (7) and (8) we get, that components of the energy-momentum tensor are
given by the Schwarzian derivatives of $A(z)$ and $B(\bar{z})$:

\[
T = \{ A; z \} = \frac{1}{2b^2} \left[ \frac{A'''}{A'} - \frac{3(A'')^2}{2(A')^2} \right], \quad (9)
\]

\[
\bar{T} = \{ B; \bar{z} \} = \frac{1}{2b^2} \left[ \frac{B'''}{B'} - \frac{3(B'')^2}{2(B')^2} \right]. \quad (10)
\]

The Schwarzian derivative is invariant under arbitrary constant Möbius transformation:

\[
\left\{ \frac{\zeta F + \beta}{\gamma F + \delta}; z \right\} = \{ F; z \}, \quad \zeta \delta - \beta \gamma = 1. \quad (11)
\]
Lagrangian of the Liouville theory with defect

Recently the action of the Liouville theory with topological defects was suggested:

\[ S_{\text{top-def}} = \frac{1}{2\pi i} \int_{\Sigma_1} \left( \partial \phi_1 \bar{\partial} \phi_1 + \mu \pi e^{2b \phi_1} \right) d^2 z + \]

\[ \frac{1}{2\pi i} \int_{\Sigma_2} \left( \partial \phi_2 \bar{\partial} \phi_2 + \mu \pi e^{2b \phi_2} \right) d^2 z \]

\[ + \int_{\partial \Sigma_1} \left[ -\frac{1}{2\pi} \phi_2 \partial_\tau \phi_1 + \frac{1}{2\pi} \Lambda \partial_\tau (\phi_1 - \phi_2) + \right. \]

\[ \left. \frac{\mu}{2} e^{(\phi_1 + \phi_2 - \Lambda) b} - \frac{1}{\pi b^2} e^{\Lambda b} (\cosh(\phi_1 - \phi_2) b - \kappa) \right] d_\tau \]

Here \( \Sigma_1 \) is the upper half-plane \( \sigma = \text{Im} z \geq 0 \) and \( \Sigma_2 \) is the lower half-plane \( \sigma = \text{Im} z \leq 0 \). The defect is located along their common boundary, which is the real axis \( \sigma = 0 \) parametrized by \( \tau = \text{Re} z \). Note that \( \Lambda(\tau) \) here is an additional field associated with the defect itself. The ac-
tion (12) yields the following defect equations of motion at $\sigma = 0$:

\[ \bar{\partial} (\phi_1 - \phi_2) = \pi \mu b e^{b(\phi_1 + \phi_2)} e^{-\Lambda b}, \quad (13) \]

\[ \partial (\phi_1 - \phi_2) = \frac{2}{b} e^{\Lambda b} \left( \cosh(\phi_1 - \phi_2)b - \kappa \right). \quad (14) \]

\[ \partial (\phi_1 + \phi_2) - \partial_\tau \Lambda = \frac{2}{b} e^{\Lambda b} \sinh(b(\phi_1 - \phi_2)). \quad (15) \]

Requiring the defect equations of motion to hold for every $\sigma$ brings additionally to the condition, that $\Lambda$ is restriction to the real axis of a holomorphic field

\[ \bar{\partial} \Lambda = 0. \quad (16) \]

This condition allows to rewrite (15) in the form

\[ \partial (\phi_1 + \phi_2 - \Lambda) = \frac{2}{b} e^{\Lambda b} \sinh(b(\phi_1 - \phi_2)). \quad (17) \]
The system of the defect equations of motion (13)-(17) guarantees that both components of the energy-momentum tensor are continuous across the defects and therefore describes topological defects:

\[ -(\partial \phi_1)^2 + b^{-1} \partial^2 \phi_1 = -(\partial \phi_2)^2 + b^{-1} \partial^2 \phi_2, \quad (18) \]

\[ -(\bar{\partial} \phi_1)^2 + b^{-1} \bar{\partial}^2 \phi_1 = -(\bar{\partial} \phi_2)^2 + b^{-1} \bar{\partial}^2 \phi_2. \quad (19) \]

Now we will present the general solution for defect equations of motion (13)-(17).

On the one hand since the defect is topological both components of the energy-momentum tensor are equal being computed in terms of \( \phi_1 \) or \( \phi_2 \). On the other hand each component of the energy-momentum tensor is given by the
Schwarzian derivative, which is invariant under the Möbius transformation. This naturally leads to the following solution:

\[ \phi_1 = \frac{1}{2b} \log \left( \frac{1}{\pi \mu b^2} \frac{\partial A \bar{\partial} B}{(A + B)^2} \right), \quad (20) \]

\[ \phi_2 = \frac{1}{2b} \log \left( \frac{1}{\pi \mu b^2} \frac{\partial C \bar{\partial} D}{(C + D)^2} \right), \quad (21) \]

where

\[ C = \frac{\zeta A + \beta}{\gamma A + \delta} \quad \text{and} \quad D = \frac{\zeta' B + \beta'}{\gamma' B + \delta'}. \quad (22) \]

Remembering that \( \phi_2 \) is invariant under the simultaneous Möbius transformation (6) of \( C \) and \( D \), we can set \( B = D \). Therefore without losing generality we can look for a solution in the form:
\[ \phi_1 = \frac{1}{2b} \log \left( \frac{1}{\pi \mu b^2} \frac{\partial A \bar{\partial} B}{(A + B)^2} \right), \quad (23) \]

\[ \phi_2 = \frac{1}{2b} \log \left( \frac{1}{\pi \mu b^2} \frac{\partial C \bar{\partial} B}{(C + B)^2} \right), \quad (24) \]

where

\[ C = \frac{\zeta A + \beta}{\gamma A + \delta}. \quad (25) \]

Substituting (23) and (24) in (13) we find that it is satisfied with

\[ e^{-\Lambda b} = \frac{A - C}{\sqrt{\partial A \partial C}}. \quad (26) \]

Since \( A \) and \( C \) are holomorphic functions, \( \Lambda \) is holomorphic as well, as it is stated in (16).

It is straightforward to check that (17) is satisfied as well with \( \phi_1, \phi_2 \) and \( \Lambda \) given by (23),
(24) and (26) respectively. And finally inserting (23), (24) and (26) in (14) we see that it is also fulfilled with

\[ \kappa = \frac{\zeta + \delta}{2}. \]  

(27)
Defects in Quantum Liouville

Review of quantum Liouville

Liouville field theory is a conformal field theory enjoying the Virasoro algebra

\[ [L_m, L_n] = (m-n)L_{m+n} + \frac{c_L}{12}(n^3-n)\delta_{n,-m}, \quad (28) \]

with the central charge

\[ c_L = 1 + 6Q^2. \quad (29) \]

Primary fields \( V_\alpha \) in this theory, which are associated with exponential fields \( e^{2\alpha \varphi} \), have conformal dimensions

\[ \Delta_\alpha = \alpha(Q - \alpha). \quad (30) \]

The fields \( V_\alpha \) and \( V_{Q-\alpha} \) have the same conformal dimensions and represent the same primary field,
i.e. they are proportional to each other:

\[ V_\alpha = S(\alpha)V_{Q-\alpha}, \quad (31) \]

with the reflection function

\[
S(\alpha) = \left( \frac{\pi \mu \gamma(b^2)}{b^2} \right)^{b-1}(Q-2\alpha) \\
\times \frac{\Gamma(1-b(Q-2\alpha))\Gamma(-b^{-1}(Q-2\alpha))}{\Gamma(b(Q-2\alpha))\Gamma(1+b^{-1}(Q-2\alpha))}. \quad (32)
\]

Two-point functions of Liouville theory are given by the reflection function (32):

\[
\langle V_\alpha(z_1, \bar{z}_1)V_\alpha(z_2, \bar{z}_2) \rangle = \frac{S(\alpha)}{(z_1 - z_2)^{2\Delta_\alpha}(\bar{z}_1 - \bar{z}_2)^{2\Delta_\alpha}}. \quad (33)
\]

Introducing ZZ function:

\[
W(\alpha) = -\frac{2^{3/4}e^{3i\pi/2}(\pi \mu \gamma(b^2))^{-\frac{(Q-2\alpha)}{2b}}}{\Gamma(1-b(Q-2\alpha))\Gamma(1-b^{-1}(Q-2\alpha))} \pi(Q-2\alpha), \quad (34)
\]
the two-point function can be compactly written as

\[ S(\alpha) = \frac{W(Q - \alpha)}{W(\alpha)}. \]  

(35)

Another useful property of ZZ function is

\[ W(Q - \alpha)W(\alpha) = -i2\sqrt{2}\sin\pi b^{-1}(2\alpha - Q)\sin\pi b(2\alpha - Q). \]  

(36)

The spectrum of the Liouville theory has the form

\[ \mathcal{H} = \int_{0}^{\infty} dP \, R_{\frac{Q}{2} + iP} \otimes R_{\frac{Q}{2} + iP}, \]  

(37)

where \( R_{\alpha} \) is the highest weight representation with respect to the Virasoro algebra.
Defects in quantum Liouville

Topological defects are intertwining operators $X$ commuting with the Virasoro generators

$$[L_n, X] = [ar{L}_n, X] = 0.$$ \hfill (38)

Such operators have the form

$$X = \int_{Q/2 + i\mathbb{R}} d\alpha \, \mathcal{D}(\alpha) \, \mathbb{P}^\alpha,$$ \hfill (39)

where $\mathbb{P}^\alpha$ are projectors on a subspace $R_\alpha \otimes R_\alpha$:

$$\mathbb{P}^\alpha = \sum_{N,M} (|\alpha, N\rangle \otimes |\alpha, M\rangle)(\langle \alpha, N| \otimes \langle \alpha, M|).$$ \hfill (40)

Here $|\alpha, N\rangle$ and $|\alpha, M\rangle$ are vectors of orthonormal bases of left and right copy of $R_\alpha$ respectively.

The eigenvalues $\mathcal{D}(\alpha)$ can be determined via the two-point functions computed in the presence of
The defect two-point functions have the form

\[
\langle V_\alpha(z_1, \bar{z}_1) X V_\alpha(z_2, \bar{z}_2) \rangle = \frac{D(\alpha)S(\alpha)}{(z_1 - z_2)^{2\Delta_\alpha}(\bar{z}_1 - \bar{z}_2)^{2\Delta_\alpha}}
\]

(41)

and therefore for \( D_s(\alpha) \) one can write using (35) and (36)

\[
D_s(\alpha) = -\frac{2^{1/2}i \cosh(2\pi s(2\alpha - Q))}{\cosh(2\pi s(2\alpha - Q))} \frac{S(\alpha)W^2(\alpha)}{2 \sin \pi b^{-1}(2\alpha - Q) \sin \pi b(2\alpha - Q)}.
\]
Semiclassical limits

Heavy asymptotic limit

Let us consider the action (3) for the rescaled variable $\varphi = 2b\phi$

$$S = \frac{1}{8\pi ib^2} \int \left( \partial \varphi \bar{\partial} \varphi + 4\lambda e^\varphi \right) d^2 z,$$

where $\lambda = \pi \mu b^2$.

This form shows that $b^2$ plays in the Liouville theory the role of the Planck constant, and one can study semiclassical limit taking the limit $b \to 0$, in such a way that $\lambda$ is kept fixed.

Let us consider correlation functions in the path
integral formalism:

\[
\langle V_{\alpha_1}(z_1, \bar{z}_1) \cdots V_{\alpha_n}(z_n, \bar{z}_n) \rangle = \int \mathcal{D}\varphi \ e^{-S} \prod_{i=1}^{n} \exp \left( \frac{\alpha_i \varphi(z_i, \bar{z}_i)}{b} \right).
\]

(45)

We would like to calculate this integral in the semiclassical limit \( b \to 0 \) using the method of steepest descent, and we should decide how \( \alpha_i \) scales with \( b \). Since \( S \) scales like \( b^{-2} \), for operators to affect the saddle point, we should take \( \alpha_i = \eta_i / b \), with \( \eta_i \) fixed. The conformal weights \( \Delta_\alpha = \eta(1 - \eta) / b^2 \) scale like \( b^{-2} \) as well. This is the heavy asymptotic limit.

We see from (45) that in the semiclassical limit the correlation function is given by \( e^{-S_{\text{cl}}} \) where, at least naively, in a sense which will be clarified
below, \( S_{\text{cl}} \) is the action
\[
S = \frac{1}{8\pi ib^2} \int \left( \partial \varphi \bar{\partial} \varphi + 4\lambda e^{\varphi} \right) d^2 z + \sum_{i=1}^{n} \frac{\eta_i}{b^2} \varphi(z_i, \bar{z}_i),
\]
evaluated on the solution of its equation of motion:
\[
\partial \bar{\partial} \varphi = 2\lambda e^{\varphi} - 4\pi \sum_{i=1}^{n} \eta_i \delta^2(z - z_i).
\]
Assuming that in the vicinity of the insertion point \( z_i \), one can ignore the exponential term we get that in the neighborhood of the point \( z_i \) \( \varphi \) has the following behavior
\[
\varphi(z, \bar{z}) = -4\eta_i \log |z - z_i| + X_i \text{ as } z \to z_i.
\]
One can insert this solution back into the equation of motion to check, if indeed the exponential term is subleading. We find, that this hap-
pens when
\[ \text{Re } \eta_i < \frac{1}{2}. \quad (49) \]

This constraint is known as Seiberg bound. It is the semiclassial version of the quantum condition (31) stating that \( V_\alpha \) and \( V_{Q-\alpha} \) represent the same quantum operator. Either \( \alpha \) or \( Q-\alpha \) always obey the Seiberg bound.

Remembering that in the Liouville theory we have also background charge at infinity, conditions (48) should be complemented by the behavior at the infinity:

\[ \varphi(z, \bar{z}) = -2 \log |z|^2 \text{ as } |z| \to \infty. \quad (50) \]

If one tries naively to evaluate the action (46) on a solution obeying (48), we find that it diverges.
Therefore we should consider a regularized action. It was constructed by ZZ:

\[ b^2 S^{\text{reg}} = \frac{1}{8\pi i} \int_{D-\bigcup_i d_i} \left( \partial \bar{\varphi} \bar{\partial} \varphi + 4\lambda e^\varphi \right) d^2 z \]

\[ + \frac{1}{2\pi} \oint_{\partial D} \varphi d\theta + 2 \log R \]

\[ - \sum_{i=1}^{n} \left( \frac{\eta_i}{2\pi} \oint_{d_i} \varphi d\theta_i + 2\eta_i^2 \log \epsilon_i \right). \]

Here \( D \) is a disc of radius \( R \), \( d_i \) is a disc of radius \( \epsilon_i \) around \( z_i \). The action (51) satisfies the equation

\[ \frac{\partial}{\partial \eta_i} b^2 S^{\text{reg}} = -X_i, \]

where \( X_i \) is defined by the boundary condition (48).
Defects in the heavy asymptotic limit

Heavy asymptotic limit of the correlation functions

In this section we consider the heavy asymptotic limit of two-point functions in the presence of defects (42). Now we should compute the inverse $ZZ$ function (34) and the factor $\cosh(2\pi s(2\alpha - Q))$ in the limit $b \to 0$, setting $\alpha = \frac{\eta}{b}$, and $s = \frac{\sigma}{b}$.

In the heavy asymptotic limit we should keep only terms having the form $\sim e^{1/b^2}$.

Using properties of $\Gamma$ functions we obtain

$$W_{\alpha = \frac{\eta}{b}}^{-1} \to \lambda \frac{1 - 2\eta}{2b^2} \exp \left( \frac{2\eta - 1}{b^2} \left[ \log(1 - 2\eta) - 1 \right] \right).$$

(53)
Setting $\alpha = \frac{\eta}{b}$ and $s = \frac{\sigma}{b}$ we easily obtain:

$$\cosh 2\pi s(2\alpha - Q) \rightarrow e^{\frac{2\pi b^2}{b^2} |\sigma|(1-2\eta)}. \quad (54)$$

Now we are position to write down the limiting form of the defects correlation functions.

Inserting (53), (54) in (42) we can write in the heavy asymptotic limit

$$\langle V_\alpha(z_1, \bar{z}_1) XV_\alpha(z_2, \bar{z}_2) \rangle \sim \quad (55)$$

$$\left( z_1 - z_2 \right)^{-2\eta(1-\eta)/b^2} \left( \bar{z}_1 - \bar{z}_2 \right)^{-2\eta(1-\eta)/b^2} \times \lambda^{\frac{1-2\eta}{b^2}} \exp \left( \frac{4\eta - 2}{b^2} \left[ \log(1 - 2\eta) - 1 \right] \right) e^{\frac{2\pi b^2}{b^2} |\sigma|(1-2\eta)}. $$

Finally we get

$$\langle V_\alpha(z_1, \bar{z}_1) XV_\alpha(z_2, \bar{z}_2) \rangle \sim \exp \left( -S_{\text{def}}^\text{eff} \right), \quad (56)$$
where

\[ b^2 S^{\text{def}} = 4\eta (1 - \eta) \log |z_1 - z_2| - (1 - 2\eta) \log \lambda - \]

\[ (4\eta - 2) \log (1 - 2\eta) + (4\eta - 2) - 2\pi |\sigma| (1 - 2\eta). \]
Evaluation of the action for classical solutions

According to general prescription of the semi-classical heavy asymptotic limit, we should find solutions of the Liouville equation, satisfying the defect equations of motion and possessing the logarithmic singularities (48) at points $z_1$ and $z_2$. The form of the solution of the defect equations of motion (23) and (24) implies that we should find functions $A(z)$, $C(z)$ and $B(\bar{z})$ in such a way that $\phi_1$ has a logarithmic singularity at the point $z_1$ and $\phi_2$ has a logarithmic singularity at the point $z_2$. Since the energy-momentum tensor is continuous across a defect this implies that we should find solutions possessing two singular
points. Two-point solutions are well known and we can build from them the Ansatz satisfying the defect equations of motion.

To build the solution with the required singularities one should take a function $A(z)$ which is smooth holomorphic function away from $z_1$ and $z_2$. Let us take $A(z)$ as

$$A(z) = e^{2\nu_1(z - z_1)^{2\eta-1}(z - z_2)^{1-2\eta}}. \quad (58)$$

Computing the energy-momentum tensor we obtain

$$b^2 T = \frac{\eta(1-\eta)}{(z - z_1)^2} + \frac{\eta(1-\eta)}{(z - z_2)^2} - \frac{2\eta(1-\eta)}{z_1 - z_2} \left( \frac{1}{z - z_1} - \frac{1}{z - z_2} \right), \quad (59)$$

indeed possessing two singular points.

The anti-holomorphic part is:
\[ B(\bar{z}) = -(\bar{z} - \bar{z}_1)^{1-2\eta}(\bar{z} - \bar{z}_2)^{2\eta-1}, \quad (60) \]

Let us take the holomorphic part for \( \phi_2 \) as
\[ C(z) = e^{2\nu_2(z-z_1)^{2\eta-1}(z-z_2)^{1-2\eta}} = e^{2(\nu_2-\nu_1)}A(z), \quad (61) \]
and the antiholomorphic part again given by (60).

Using (27) one gets
\[ \kappa = \cosh(\nu_2 - \nu_1). \quad (62) \]
Inserting (58), (61) and (60) in (23) and (24) we obtain:

\[ e^{-\varphi_1} = \frac{\lambda}{(2\eta - 1)^2|z_1 - z_2|^2} \times (e^{\nu_1}|z - z_1|^{2\eta}|z - z_2|^{2-2\eta} - e^{-\nu_1}|z - z_1|^{2-2\eta}|z - z_2|^{2\eta})^2, \]  

\[ e^{-\varphi_2} = \frac{\lambda}{(2\eta - 1)^2|z_1 - z_2|^2} \times (e^{\nu_2}|z - z_1|^{2\eta}|z - z_2|^{2-2\eta} - e^{-\nu_2}|z - z_1|^{2-2\eta}|z - z_2|^{2\eta})^2. \]

It is easy to see that \( \varphi_1 \) and \( \varphi_2 \) given by (63) and (64) have the required singularity (48) around \( z_1 \) and \( z_2 \) respectively.
From (63) and (64) we obtain

$$\varphi_1 = -\log \lambda + 2 \log(1 - 2\eta) \quad (65)$$

$$-2 \log \left( \frac{e^{\nu_1}|z - z_1|^{2\eta}|z - z_2|^{2-2\eta}}{|z_1 - z_2|} - \frac{e^{-\nu_1}|z - z_1|^{2-2\eta}|z - z_2|^{2\eta}}{|z_1 - z_2|} \right),$$

$$\varphi_2 = -\log \lambda + 2 \log(1 - 2\eta) \quad (66)$$

$$-2 \log \left( -\frac{e^{\nu_2}|z - z_1|^{2\eta}|z - z_2|^{2-2\eta}}{|z_1 - z_2|} + \frac{e^{-\nu_2}|z - z_1|^{2-2\eta}|z - z_2|^{2\eta}}{|z_1 - z_2|} \right).$$

To evaluate the action on solutions (63), (64), we will use the strategy used in ZZ. Namely we will write the system of differential equations which this action should satisfy. The first equation is (52), which given that $\eta_1 = \eta_2 = \eta$, reads

$$b^2 \frac{\partial S_{cl}^{\text{def}}}{\partial \eta} = -X_1 - X_2. \quad (67)$$
where $X_i$ is defined in (48). The leading terms of $\varphi_1$ around $z_1$ are

$$
\varphi_1 \rightarrow -4\eta \log |z - z_1| + X_1,
$$

(68)

where

$$
X_1 = -\log \lambda + 2 \log(1 - 2\eta) - (2 - 4\eta) \log |z_1 - z_2| - 2\nu_1.
$$

(69)

The leading terms of $\varphi_2$ around $z_2$ similarly are

$$
\varphi_2 \rightarrow -4\eta \log |z - z_2| + X_2,
$$

(70)

where

$$
X_2 = -\log \lambda + 2 \log(1 - 2\eta) - (2 - 4\eta) \log |z_1 - z_2| + 2\nu_2.
$$

(71)
Inserting (69) and (71) in (67) one obtains

\[ b^2 \frac{\partial S_{\text{cl}}^{\text{def}}}{\partial \eta} = 2 \log \lambda - 4 \log(1 - 2\eta) + (72) \]

\[(4 - 8\eta) \log |z_1 - z_2| + 2(\nu_1 - \nu_2).\]

Integrating equation (72) we obtain:

\[ b^2 S_{\text{def}} = 4\eta (1 - \eta) \log |z_1 - z_2| \quad (73) \]

\[ + 2\eta \log \lambda - (4\eta - 2) \log(1 - 2\eta) + \]

\[ 4\eta - (\nu_1 - \nu_2)(1 - 2\eta) + C, \]

where \( C \) is a constant.

To fix the constant term we can directly compute the action for the Ansatz (65)-(66) with \( \eta = 0 \):
\[ \varphi_1 = - \log \lambda - \log \left( \frac{e^{\nu_1}}{|z_1 - z_2|} |z - z_2|^2 - \frac{e^{-\nu_1}}{|z_1 - z_2|} |z - z_1|^2 \right)^2, \]  
(74)

\[ \varphi_2 = - \log \lambda - \log \left( \frac{e^{\nu_2}}{|z_1 - z_2|} |z - z_2|^2 - \frac{e^{-\nu_2}}{|z_1 - z_2|} |z - z_1|^2 \right)^2. \]  
(75)

The result is

\[ b^2 S_0 = - \log \lambda - 2 - (\nu_1 - \nu_2). \]  
(76)

Comparing (76) with (73) fixes the constant \( C \):

\[ C = - \log \lambda - 2. \]  
(77)
Inserting this value of $C'$ in (73) we indeed obtain (57) if we set

\[ 2\pi\sigma = \nu_1 - \nu_2. \]  

(78)