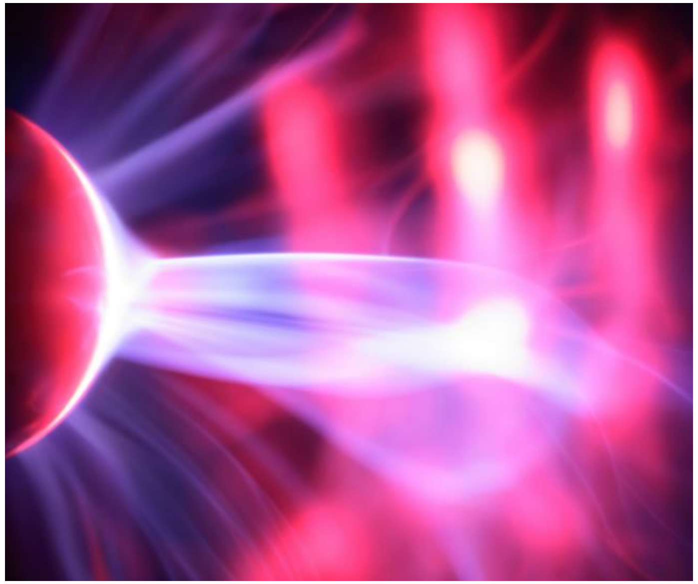
Self-focusing of an optical beam in cold plasma

Gio Chanturia

Intro

What do we have?

A laser beam (ultra short).



Cold plasma (collisionless).

Self-focusing of a beam in certain mediums

Due to non-linearity of medium, the beam focuses itself.

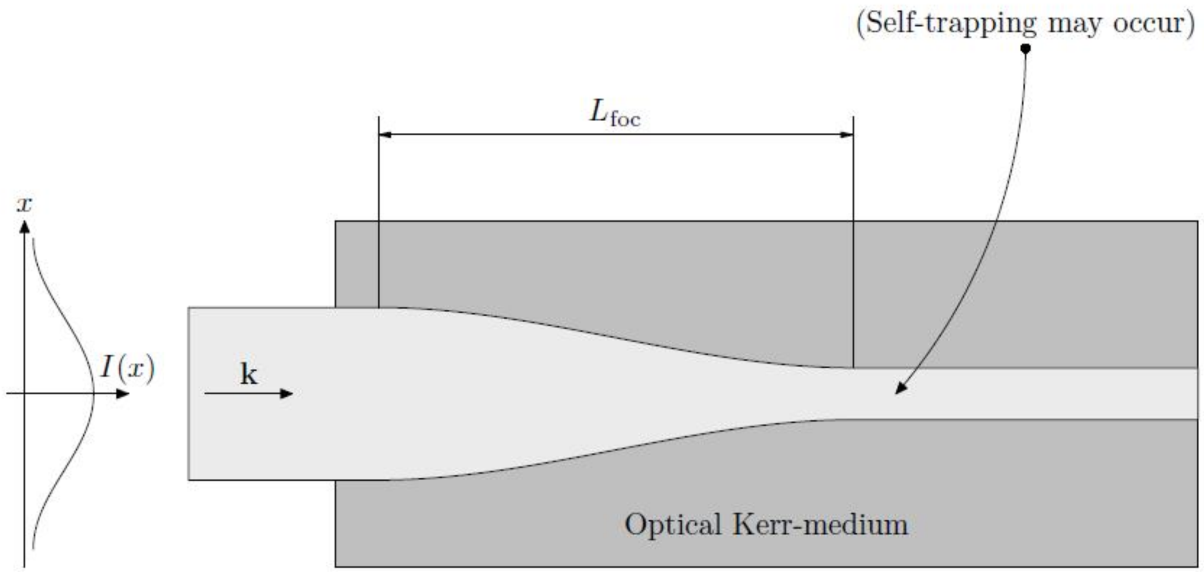


Figure 5. An illustration of the effect of self-focusing.

Cold plasma and short pulse as our model

There are reasons, we use these models:

Cool plasma model:

- Ponderomotive effect;
  - Due to electromagnetic field.
- Relativistic effect;
  - Due to free electrons in plasma.
- NO ~~thermal effect~~.

Short laser pulse model:

- Short time scale;
  - No self-focusing process for quasineutral plasma.
- Massive ions do not have time to respond and therefore stay immobile.

Describing our system mathematically

What do we need to describe our system?

Maxwell's equations and equation of motion for a relativistic electrons:

Electron velocity:

$$\boldsymbol{v} = \frac{\boldsymbol{P}}{m\gamma} = \frac{\epsilon}{mc}\frac{\boldsymbol{A}}{\sqrt{1+I_n}}$$

Plasma current:

$$\boldsymbol{J} = -en_e\boldsymbol{v} = -\frac{\omega_p^2}{4\pi c}\frac{N_e}{\sqrt{1+I_n}}\boldsymbol{A}$$

$$N_e = 1 + \frac{\delta n_e}{n_o}$$

Amplitude:

$$\boldsymbol{A} = a_n(\boldsymbol{r},t)e^{i(k_0 z - \omega_0 t - \psi(\boldsymbol{r},t))}(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})$$

Assumptions to deal with our calculations

Assumptions for amplitude and phase equations:

$$(x, y, z) \rightarrow (r, \theta, z)$$

Firstly, as we deal with axial symmetry.

That is, when none of the functions depend on θ and we're left with only two variables:

$$a(r, z) = a(r)$$

$$\psi(r, z) = f(z) + g(r)$$

Secondly, we assume, that amplitude doesn't vary towards z direction.

We allow phase to modulate and seek for solution as a sum of individual functions.

The first simplification

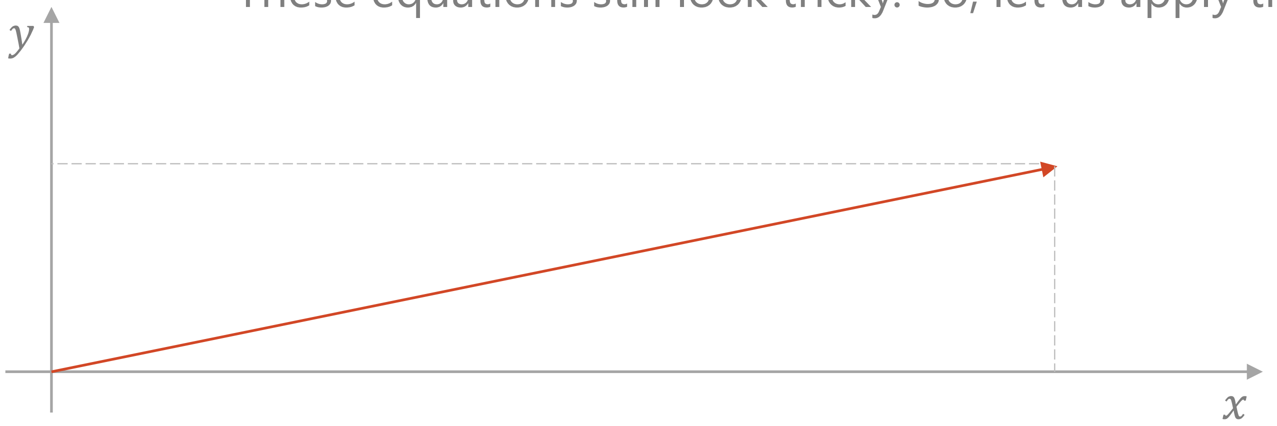
Applying previous assumptions and separating variables, we get:

$$\frac{1}{a}\frac{d^2a}{dr^2} - \left(\frac{dg}{dr}\right)^2 - \frac{1}{{\lambda_c}^2}\frac{N_e}{\sqrt{1+a^2}} = C_1$$

{separation constant}

$$\frac{d^2g}{dr^2} + \frac{1}{a^2}\frac{d(a^2)}{dr}\frac{dg}{dr} = 0$$

These equations still look tricky. So, let us apply the slab limit:



$$y \to 0$$
$$r \to x$$

Slab limit results

Within the slab limit we have:

$$\frac{1}{a}\frac{d^2a}{dx^2} - \frac{C_4^2}{a^4} - \frac{1}{{\lambda_c}^2}\frac{N_e}{\sqrt{1+a^2}} = C_1$$

$$N_e = 1 + {\lambda_c}^2 \frac{d^2}{dx^2}\sqrt{1+a^2}$$

Which combines into:

$$\frac{1}{a}\frac{d^2a}{dx^2} - \frac{C_4^2}{a^4} - \frac{1}{{\lambda_c}^2\sqrt{1+a^2}} - \frac{1}{\sqrt{1+a^2}}\frac{d^2}{dx^2}\sqrt{1+a^2} = C_1$$

Lagrangian analogy

The last equation can be written in a form:

$$\frac{1}{a}\frac{d^2a}{dx^2} - \frac{C_4^2}{a^4} - \frac{1}{{\lambda_c}^2\sqrt{1+a^2}} - \frac{1}{\sqrt{1+a^2}}\frac{d^2}{dx^2}\sqrt{1+a^2} - C_1 = 0 \qquad \equiv F(a, a', a'')$$

We know from the least action principle, that Lagrangian of a particle is written like this:

$$L = g(a)\frac{(a')^2}{2} - V(a)$$

Which by Nother's theorem gives:

$$g(a)a'' + \frac{1}{2}\frac{dg}{da}(a')^2 - \frac{\partial V}{\partial a} = 0 \qquad \equiv G(a, a', a'')$$

Lagrangian analogy

If for some integrating factor $\mu(a)$:

$$F(a, a', a'') \cdot \mu(a) = G(a, a', a'')$$

Then we will be able to find the "potential" of amplitude and therefore describe the behavior of it.

Making calculations in this manner and flattening the metric by transformation

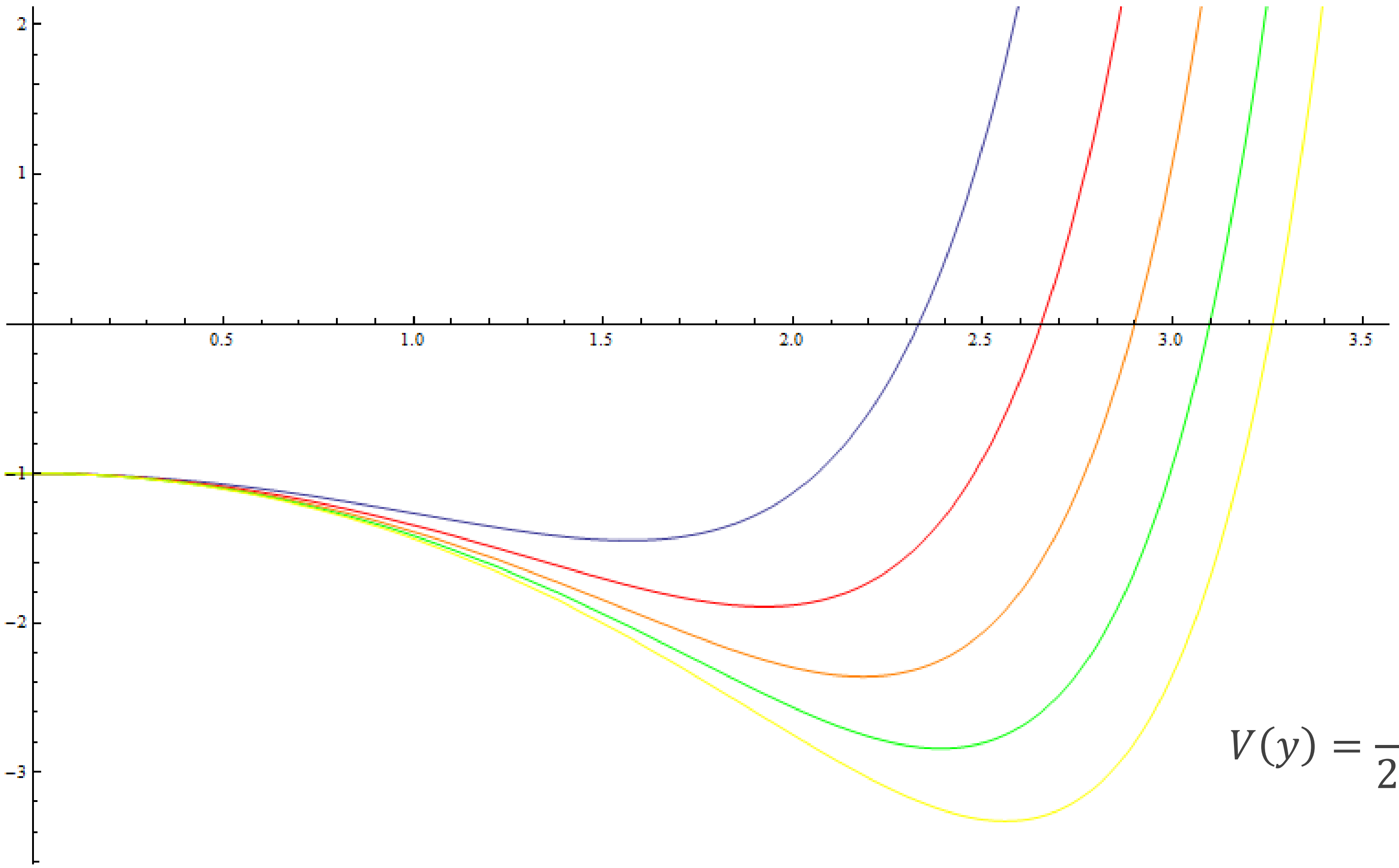
$$a = \sinh(y)$$

We obtain:

$$V(y) = \frac{\overline{C_4^2}}{2\,[\sinh(y)]^2} - \cosh(y) - \frac{\overline{C_1}}{2}[\sinh(y)]^2$$

(written in dimensionless transverse coordinate $\xi = x/\lambda_c$)

Analyzing "potential"

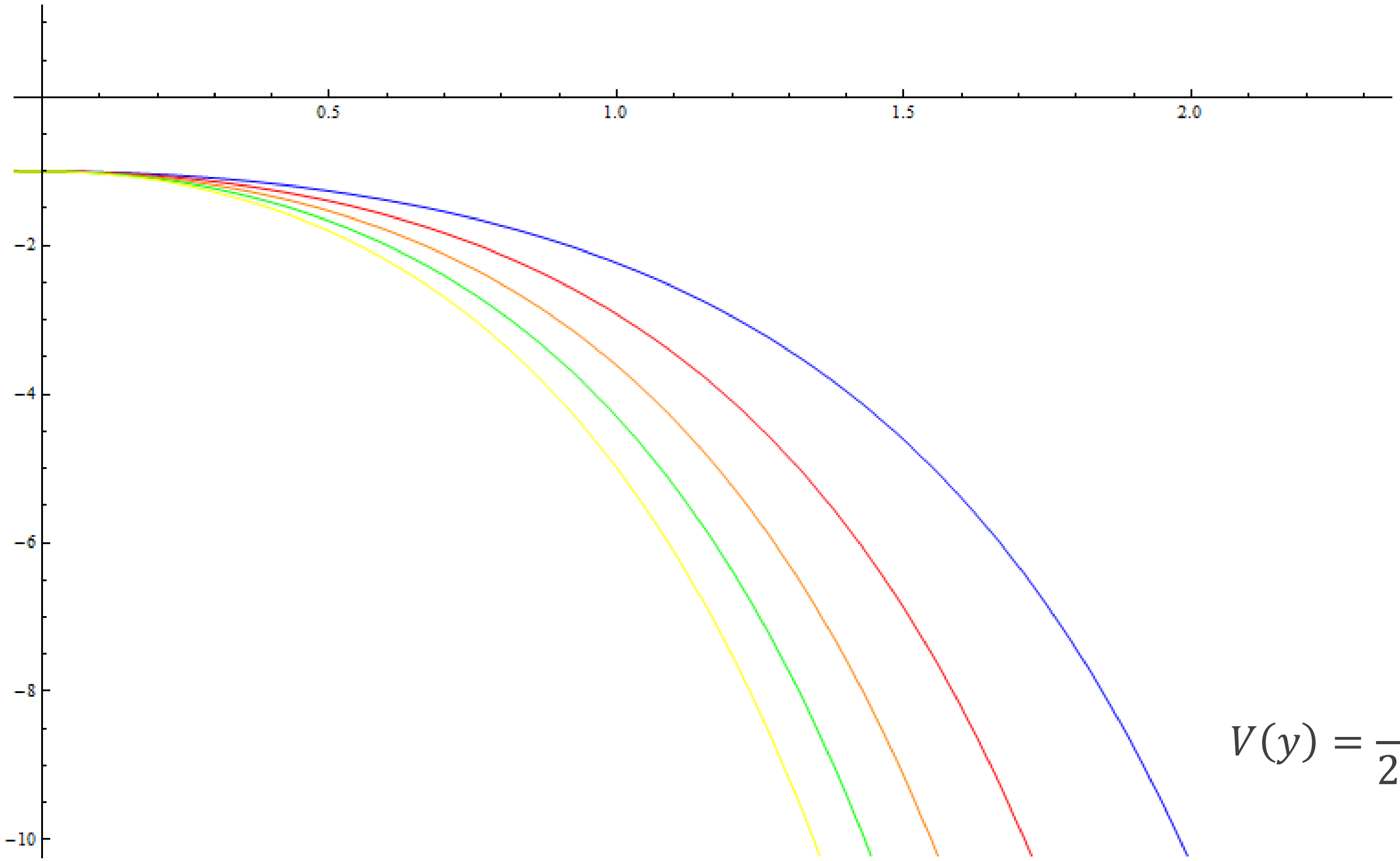


$$\overline{C_4^2} = 0$$

$$-1 < \overline{C_1} < 0$$

$$V(y) = \frac{\overline{C_4^2}}{2\,[\sinh(y)]^2} - \cosh(y) - \frac{\overline{C_1}}{2}\,[\sinh(y)]^2$$

Analyzing "potential"

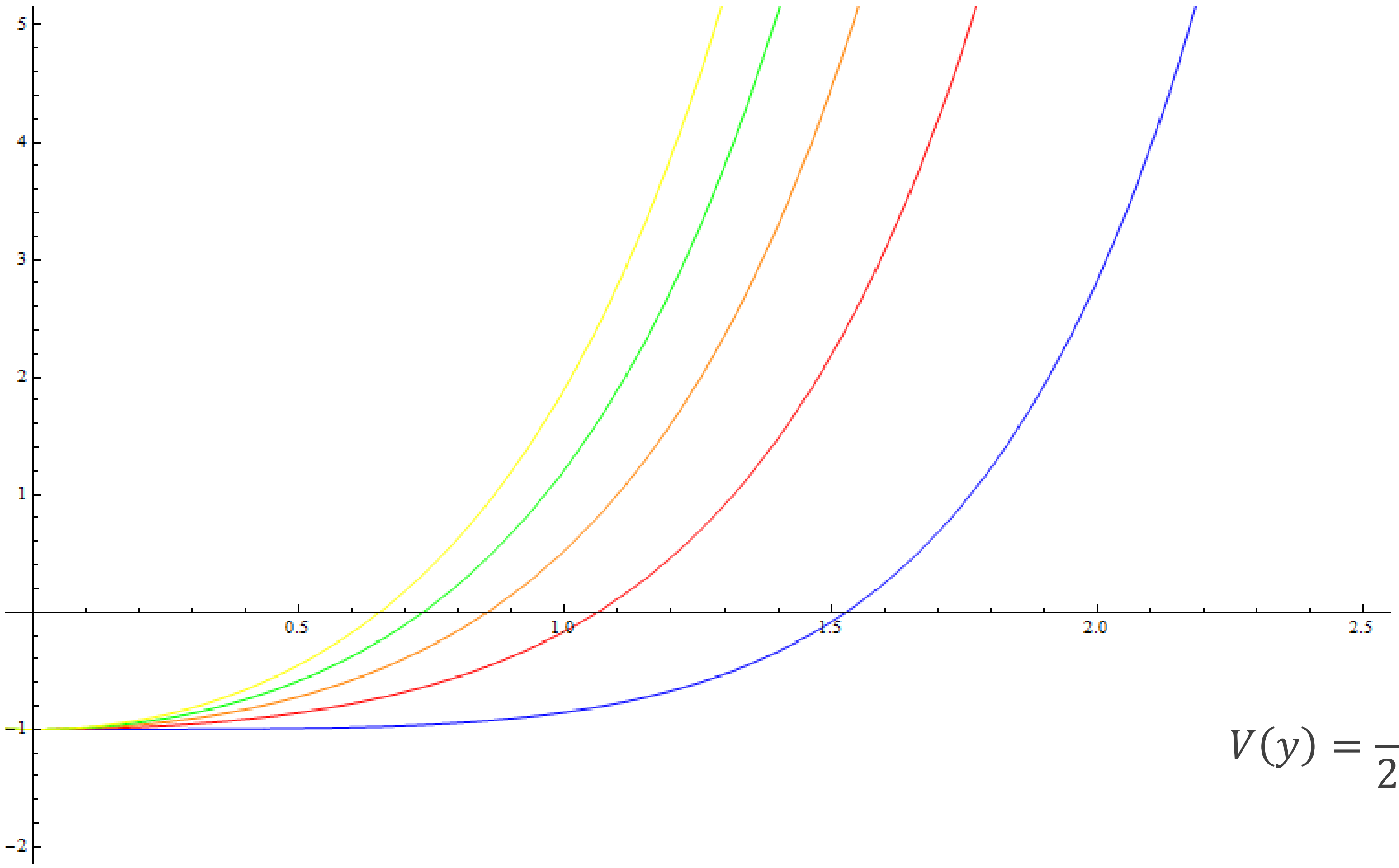


$$\overline{C_4^2} = 0$$

$$\overline{C_1} > 0$$

$$V(y) = \frac{\overline{C_4^2}}{2\,[\sinh(y)]^2} - \cosh(y) - \frac{\overline{C_1}}{2}[\sinh(y)]^2$$

Analyzing “potential”



$$\overline{C_4^2} = 0$$

$$-1 > \overline{C_1}$$

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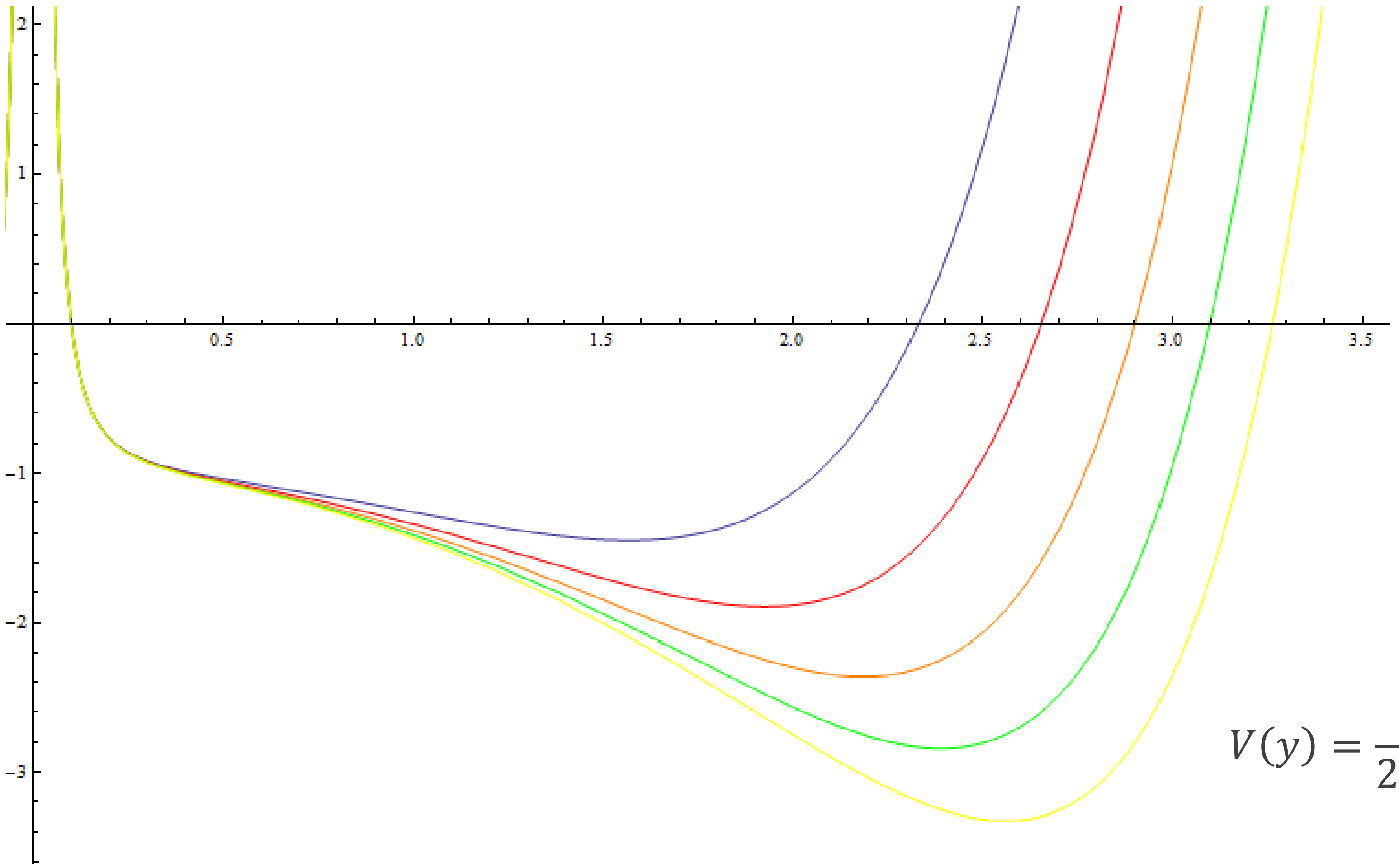
Analyzing "potential"



$$1 \gg \overline{C_4^2} > 0$$

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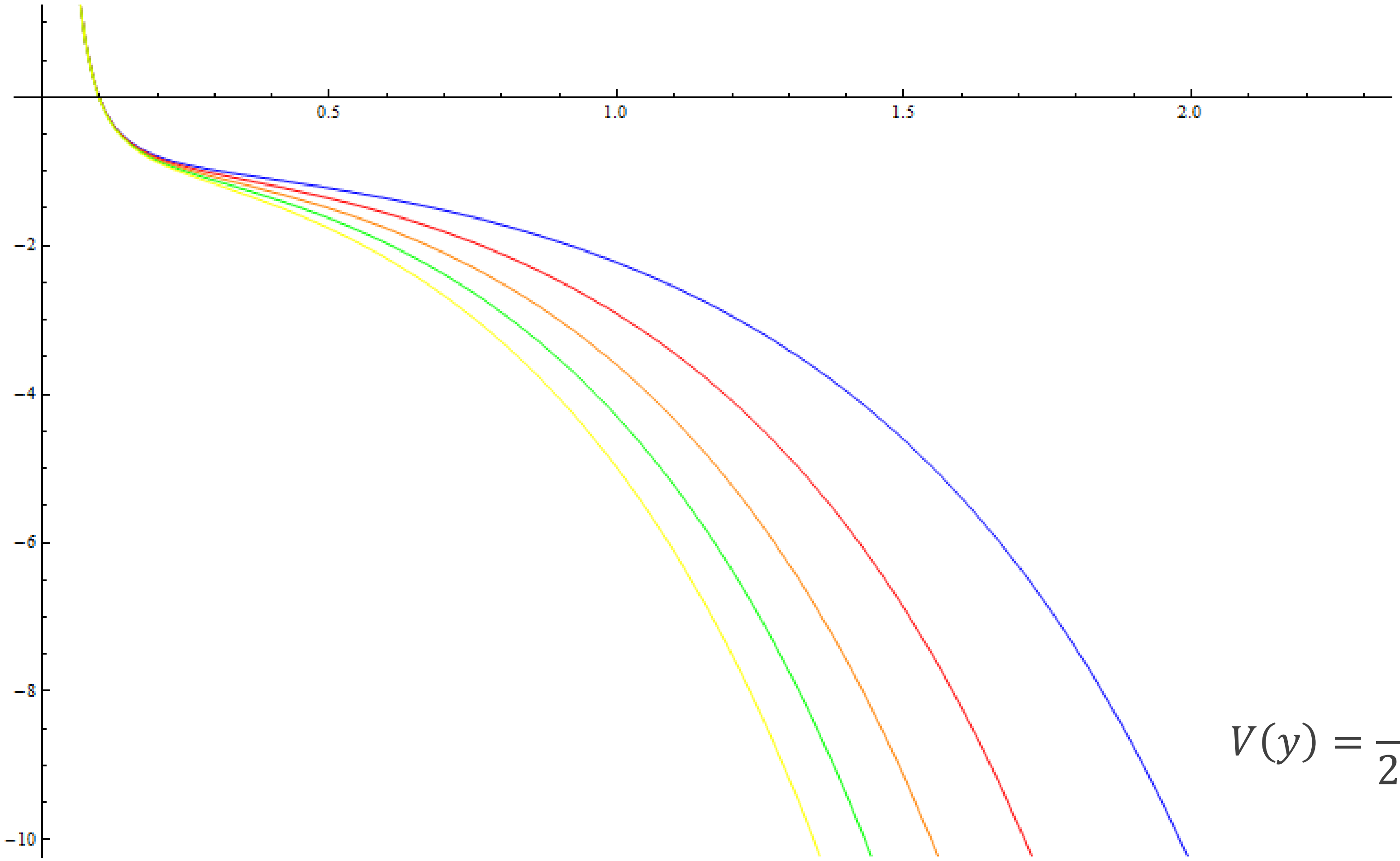

$$1 \gg \overline{C_4^2} > 0$$

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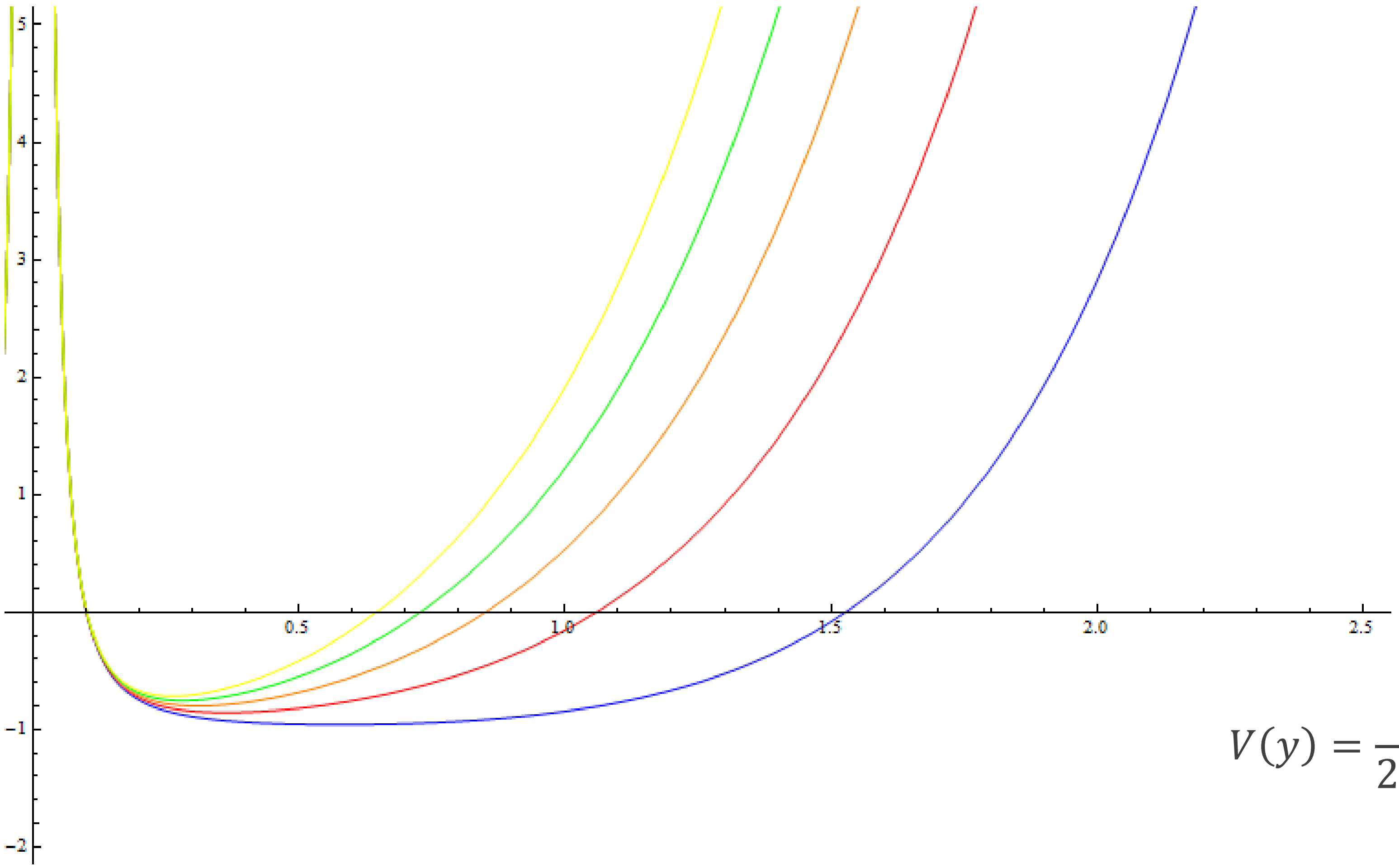
$$V(y) = \frac{\overline{C_4^2}}{2\,[\sinh(y)]^2} - \cosh(y) - \frac{\overline{C_1}}{2}[\sinh(y)]^2$$

Analyzing "potential"



$1 \gg \overline{C_4^2} > 0$

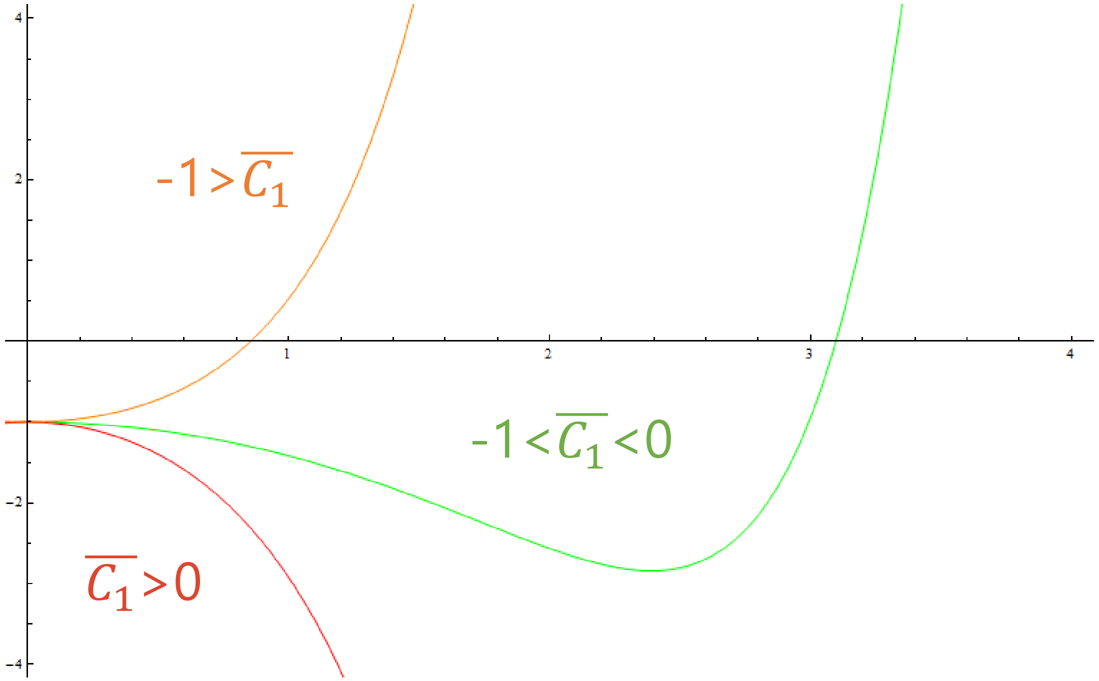
$-1 > \overline{C_1}$

$$V(y) = \frac{\overline{C_4^2}}{2\,[\sinh(y)]^2} - \cosh(y) - \frac{\overline{C_1}}{2}[\sinh(y)]^2$$

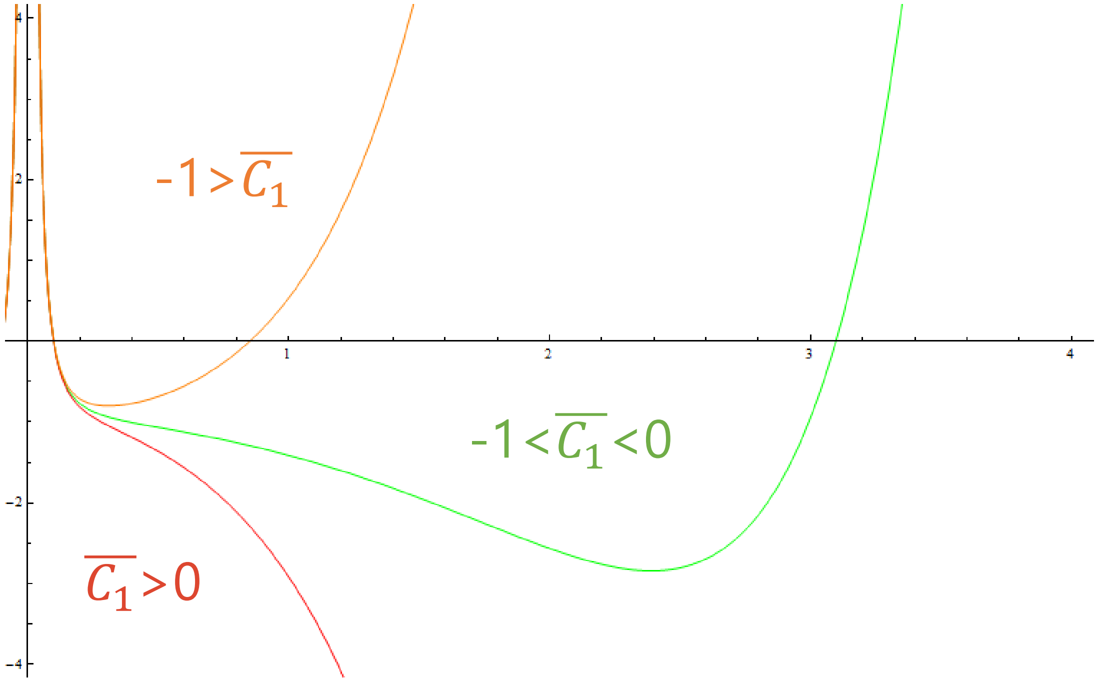
Analyzing "potential"

Summary:

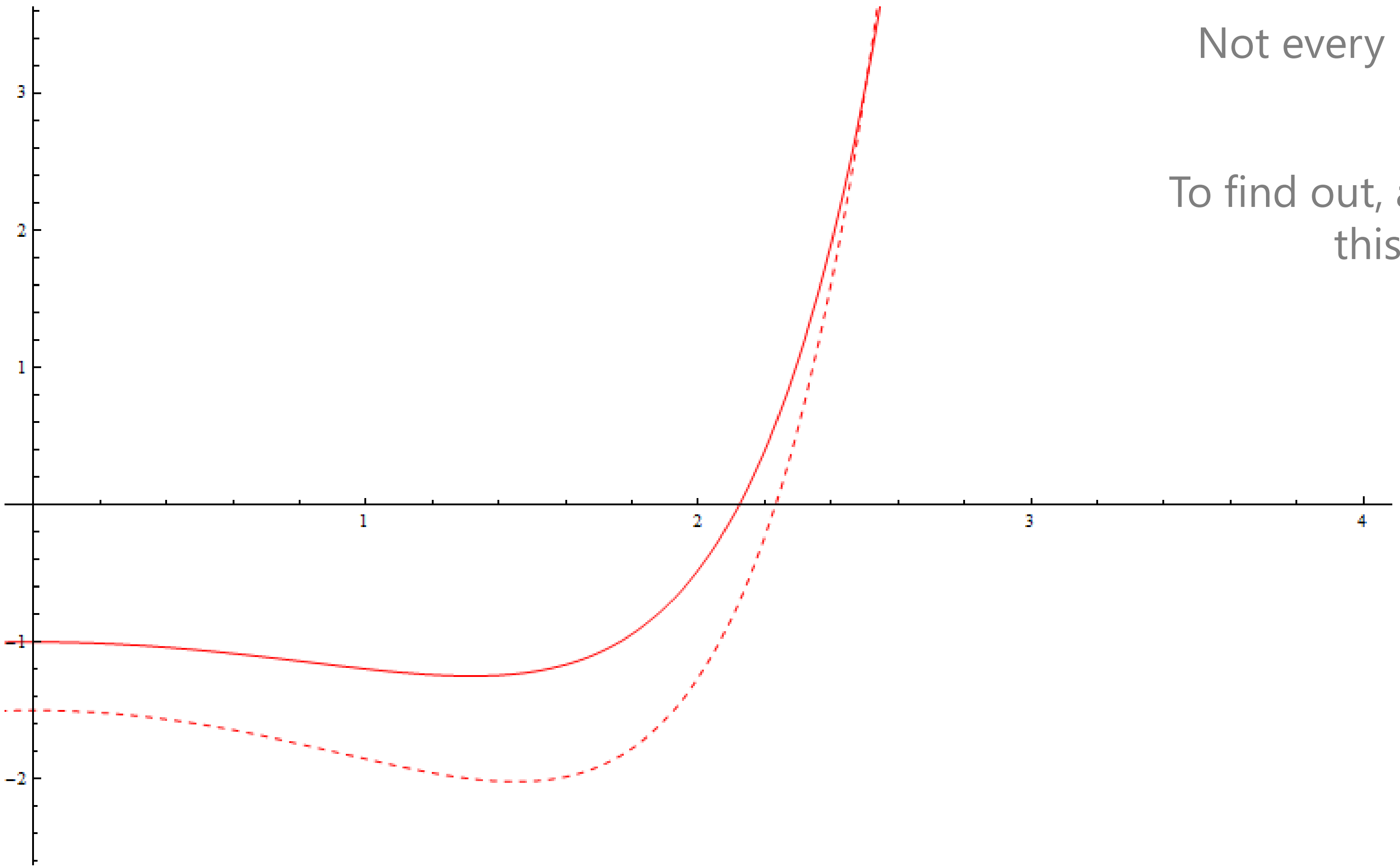
$\overline{C_4^2}=0$



$1 \gg \overline{C_4^2} > 0$



Physical values



Not every point of our potential corresponds to a physical value.

To find out, a meaningful (meaning, useful for us in this particular problem) values, we have to remember condition:

$$N_e > 0$$

Electron density can not be negative.

$$V(y) = -\cosh(y) - \frac{\overline{C_1}}{2}[\sinh(y)]^2$$

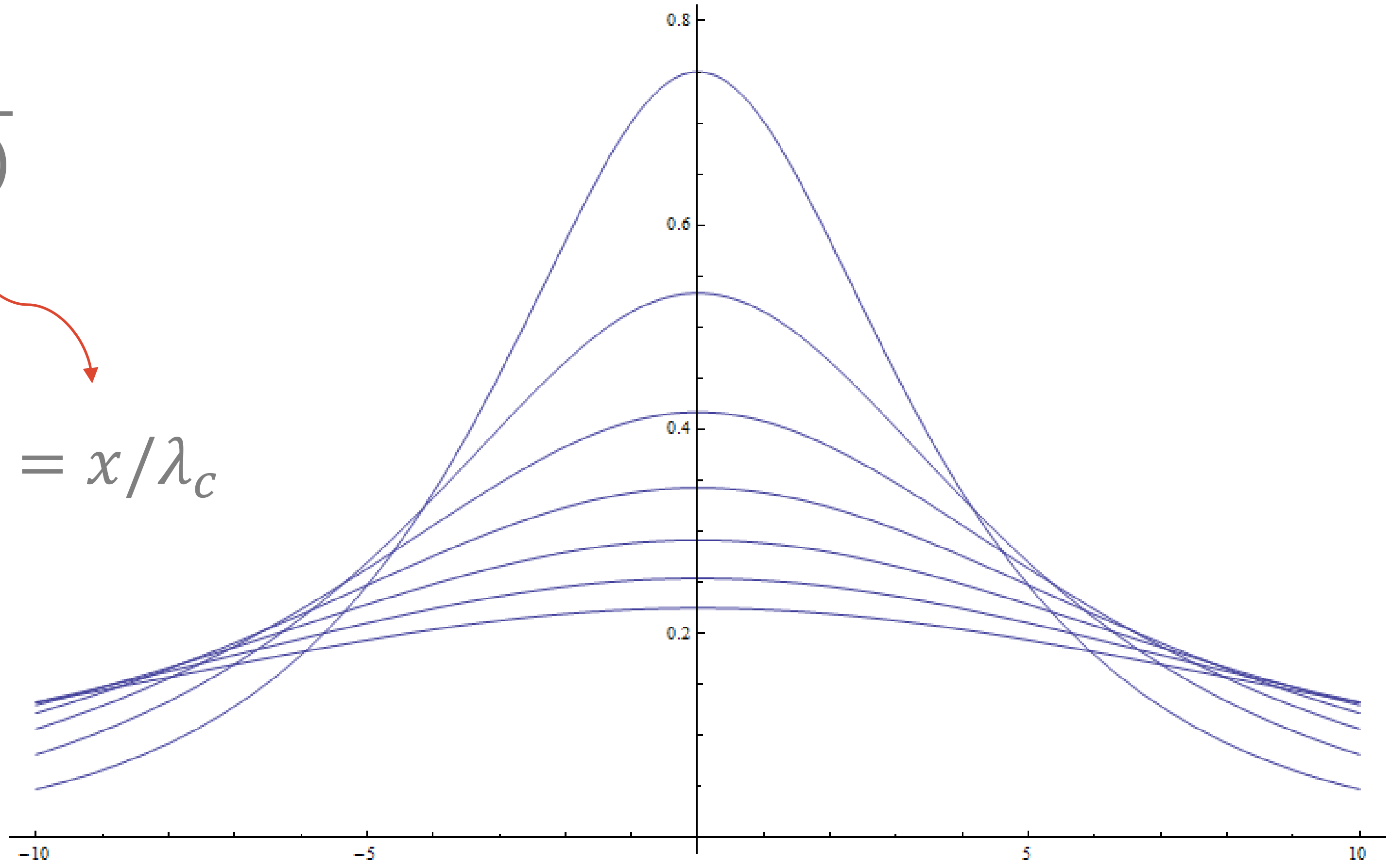
$$\varepsilon(y) > -\frac{3}{2}\cosh(y) - \overline{C_1}[\sinh(y)]^2$$

Exact solutions

$$a = \frac{2\kappa \operatorname{sech}(\kappa\xi)}{1 - \kappa^2 \operatorname{sech}^2(\kappa\xi)}$$

$$\kappa^2 = 1 + \lambda_c^2 C_1$$

$$\xi = x/\lambda_c$$



Thank you!

Sources:

- T.Kurki-Suonio, P.J. Morrison, T.Tajima – *"Self-focusing of an optical beam in plasma"*;
- Stockholm's Royal Institute of Technology – *"Nonlinear Optics 5A5513 (2003)"*;
- Wolfram's Mathematica (plots);