Dynamics of AdS particle

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Contents

Introduction

We describe classical and quantum dynamics of AdS particle.

On the classical level we analyze the action of AdS particle and construct the dynamical integrals related to the isometry group.

Using these dynamical integrals we describe particle trajectories without solving dynamical equations.

To quantize the system we use the static gauge.

Quantum calculations are performed for a particle in AdS_2.

We construct the operators for the dynamical integrals, check the algebra of their commutators and calculate the Casimir number. The energy spectrum and the corresponding eigenfunctions are obtained in the coordinate representation.

Particle dynamics in Minkowski space

The action of a relativistic particle of mass m

$$S = -m \int \sqrt{\eta_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}} \;\; d\tau$$

$\eta_{\mu\nu}$ is the metric tensor of $(N+1)$-dimensional Minkowski space with the signature $(+, -, -, ..., -)$.

The action is invariant under the Poincare transformations

$$x^{\mu} \rightarrow \Lambda^{\mu}{}_{\nu}\, x^{\mu} + a^{\mu}$$

and the reparametrizations

$$\tau \rightarrow f(\tau)$$

Dynamical integrals

The Poincare symmetry provides the dynamical integrals

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -m \frac{\dot{x}_\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}}$$

$$M_{\mu\nu} = p_\mu x_\nu - p_\nu x_\mu$$

The Poisson brackets of $p_\mu, M_{\mu\nu}$ realize the Poincare algebra

$$\{p_\mu, p_\nu\} = 0$$

$$\{M_{\mu\nu}, p_\sigma\} = \eta_{\mu\sigma} p_\nu - \eta_{\nu\sigma} p_\mu$$

$$\{M_{\mu\nu}, M_{\rho\sigma}\} = \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma}$$

M_{ab} are the generators of space rotations $(a, b) = 1, ..., N$
M_{0a} generate the Lorentz boosts.
The canonical momenta p_μ are constraint by

$$p_\mu p^\mu - m^2 = 0$$

First order action and conformal symmetry

Taking into account the constraint, one gets the action

$$S_1 = \int \left[p_\mu \dot{x}^\mu + \frac{e}{2}(p_\mu p^\mu - m^2) \right] \, d\tau$$

The variation of S_1 with respect to the canonical momenta p_μ provides $\dot{x}^\mu + e p^\mu = 0$ and the elimination of p_μ results in

$$S_2 = - \int \left[\frac{\dot{x}_\mu \dot{x}^\mu}{2e} + \frac{e m^2}{2} \right] \, d\tau$$

The limit $m \to 0$ is well defined for S_1 and S_2.
It describes the massless particle.

For the massless particle the Poincare symmetry is extended by invariance under the conformal transformations

$$\eta_{\mu\nu} \; \mapsto \; \Omega \, \eta_{\mu\nu} \qquad e \; \mapsto \; \Omega \, e$$

Gauge fixing

We first fix the gauge freedom by

$$x^0 = p_0 \tau$$

The first order action S_1 reduces to

$$S_1' = \int \left(p_a \dot{x}^a - \frac{p_0^2}{2} \right)$$

where p_0 is obtained from the constraint $p_\mu p^\mu = m^2$

$$p_0 = \sqrt{\vec{p}^{\,2} + m^2}$$

One gets $2N$ canonical coordinates (p_a, x^a).
In terms of canonical the dynamical integrals variables become

$$p_a = p_a \qquad M_{ab} = p_a x_b - p_b x_a$$

$$M_{0a} = \sqrt{\vec{p}^{\,2} + m^2}(x_a - p_a \tau)$$

Quantization

We choose p-representation.
$p_0 = \sqrt{\vec{p}^{\,2} + m^2}$ and p_a are multiplication operators.
M_{ab}, M_{0a} are first order differential operators

$$M_{ab} = ip_a\,\partial_b - ip_b\,\partial_a \quad M_{0a} = i\sqrt{\vec{p}^{\,2} + m^2}\partial_a$$

The boost operators are self-adjoint for the scalar product

$$< \Psi_2 \mid \Psi_1 >= \int \frac{d^N p}{\sqrt{\vec{p}^{\,2} + m^2}} \Psi_2^\star(p)\Psi_1(p)$$

Quantum Casimir number is $p_\mu p^\mu = m^2$
The minimal energy is m.
The massless particle corresponds to $m = 0$.

Geometry of AdS space

AdS_{N+1} is associated with the $(N+1)$-dimensional hyperboloid

$$X_0^2 + X_{0'}^2 - \sum_{n=1}^{N} X_n^2 = R^2$$

embedded in $(N+2)$-dimensional space $\mathbb{R}^{2,N}$.
Parameterizing the hyperboloid

$$X_0 = r\,\cos\theta \qquad X_{0'} = r\,\sin\theta \qquad X_n = x_n \quad (n = 1, ..., N)$$

where $\theta = x_0$ and $r = \sqrt{R^2 + x_n x_n}$.
One gets the induced metric tensor $g_{\mu\nu}$

$$g_{00} = r^2 \qquad g_{0n} = g_{n0} = 0 \qquad g_{mn} = -\delta_{mn} + \frac{x_m x_n}{r^2}$$

AdS Particle dynamics

The action of AdS particle in embedding coordinates

$$S = -\int d\tau \left[\frac{\dot{X}^A \dot{X}_A}{2e} + \frac{em^2}{2} + \frac{\lambda}{2}(X^A X_A - R^2) \right]$$

Dynamical integrals

$$J_{AB} = P_A X_B - P_B X_A$$

Notations: $J_{0n} = K_n$, $J_{0'n} = L_n$, $J_{00'} = E$.
Since θ is the time coordinate, E is the particle energy.
One finds N equations

$$E\, X_n = K_n\, X_{0'} - L_n\, X_0, \qquad (n = 1, ..., N)$$

This define a 2-dimensional plane in the embedding space $\mathbb{R}^{2,N}$.
The intersection of this plane with the hyperbola is a trajectory.

AdS Particle dynamics

The action in terms of space-time coordinates

$$S = -\int \left[\frac{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{2e} + \frac{em^2}{2} \right] \, d\tau$$

In the first order formalism the action becomes

$$S = \int \left[p_\mu \dot{x}^\mu + \frac{e}{2}(g^{\mu\nu} p_\mu p_\nu - m^2) \right] \, d\tau$$

The dynamical integrals

$$J_{AB} = p_\mu V^\mu_{AB}(x)$$

V^μ_{AB} corresponds to infinitesimal symmetry transformations.

Gauge fixing

For AdS particle we use again static gauge $x^0 = p_0\,\tau$.
After the Hamiltonian reduction we find the action

$$S = \int \left(p_a \dot{x}^a - \frac{p_0^2}{2} \right) d\tau$$

$p_0 = E$ is obtained from the mass-shell condition $g^{\mu\nu} p_\mu p_\nu = m^2$.

In AdS$_2$ with $R = 1$ one has

$$E^2 = (1 + x^2)^2\, p^2 + m^2 (1 + x^2)$$

Two other generators at $\tau = 0$ become

$$K = p\sqrt{1 + x^2} \qquad\qquad L = -\frac{E\,x}{\sqrt{1 + x^2}}$$

Canonical transformation

The canonical transformation

$$x = -\cot Q \qquad p = P \sin^2 Q$$

Simplifies the symmetry generators

$$E^2 = P^2 + \frac{m^2}{\sin^2 Q}$$

$$K = P \sin Q \qquad L = E \cos Q$$

One gets the Poisson brackets algebra

$$\{E, K\} = L \qquad \{E, L\} = -K \qquad \{K, L\} = -E$$

Quantization

To quantize the system we use the coordinate representation with

$$E^2 = -\partial^2_{QQ} + \frac{m^2}{\sin^2 Q}$$

The ground state wave function is given by

$$\psi_0(Q) = \sin^\mu Q$$

μ is the minima energy

$$\mu = \frac{1}{2} + \sqrt{m^2 + \frac{1}{4}}$$

The algebra of symmetry generators is provided by the operators

$$K = -i\sqrt{E}(\sin Q\, \partial_Q)\frac{1}{\sqrt{E}} \qquad L = \sqrt{E}\, \cos Q \,\sqrt{E}$$

Conclusions

We have quantized AdS_2 particle in the static gauge.

The construction of the isometry group generators provides the realization of the symmetry algebra.

The calculation of the Casimir number yields

$$E^2 - K^2 - L^2 = m^2$$

The massless case corresponds to $m = 0$.

The conformal group in 2-dimensions is infinite dimensional.

It is interesting to analyze whether one can realize this infinite dimensional symmetry for the AdS_2 massless particle.