Dynamics of integrable systems

Shako Kapanadze

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Introduction

Our aim is to investigate two integrable models of 2d field theory.

One model is given by Liouville equation which describes exponentially interacting field.

Another model is also nonlinear and it describes interaction of two scalar fields.

The models have conformal symmetry that becomes a basis for the integrabilty.

Our final goal is to investigate the corresponding quantum field theories.

In this presentations we are restricted to the analysis of the classical theories. We are mainly concentrated to homogeneous field configurations which provide integrable mechanical systems.

Liouville theory

The Liouville field theory is described by the dynamical equation

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) \varphi(\tau, \sigma) + 4m^2\, e^{2\varphi(\tau,\sigma)} = 0$$

where $m > 0$ is a constant and $(\sigma,\, \tau)$ are space-time coordinates. In the light-cone coordinates

$$z = \tau + \sigma \qquad \bar{z} = \tau - \sigma$$

Liouville equation becomes

$$\partial\bar{\partial}\varphi(z, \bar{z}) + m^2 e^{2\varphi(z,\bar{z})}$$

The general solution of this equation can be written as

$$\varphi(\tau, \sigma) = \frac{1}{2}\, \log \frac{A'(z)\, B'(\bar{z})}{[1 + m^2\, A(z)\, B(\bar{z})]^2}$$

where A and B are two arbitrary monotonic functions.

2d conformal transformations

The conformal transformations of space-time coordinates correspond to the transformation of the metric tensor

$$g_{\mu\nu}(x) \ \mapsto \tilde{g}_{\mu\nu}(x) = \Omega(x)\, g_{\mu\nu}(x)$$

In 2d Minkowski space with coordinates (τ, σ) the infinitesimal length is given by

$$dl^2 = d\tau^2 - d\sigma^2 = dz\, d\bar{z}$$

The conformal transformations therefore are given by

$$z \ \ \mapsto \ \ \tilde{z} = f(z) \qquad \bar{z} \ \ \mapsto \ \ \tilde{\bar{z}} = \bar{f}(\bar{z})$$

where the functions $f(z)$ and $\bar{f}(\bar{z})$ are assumed monotonic

$$f'(z) > 0 \qquad \bar{f}'(\bar{z}) > 0$$

Conformal symmetry of Liouville equation

Liouville equation is invariant under the following conformal transformations

$$\varphi(z,\bar{z}) \ \mapsto \tilde{\varphi}(z,\bar{z}) = \varphi\left(f(z), \bar{f}(\bar{z})\right) + \frac{1}{2}\log f'(z)\bar{f}'(\bar{z})$$

Conformal symmetry of Liouville equation provides the general solution.

If τ and σ are on the strip $\sigma \in (0,\pi)$, $\tau \in \mathbb{R}^1$, then

$$\bar{f}(\tau) = f(\tau) \quad \text{and} \quad f(\tau + 2\pi) = f(\tau) + 2\pi$$

For the periodic boundary conditions $f(z)$ and $\bar{f}(\bar{z})$ are independent.

Chiral structure

One can check that the fields

$$T = (\partial\varphi) - \partial^2\varphi \qquad \bar{T} = (\bar{\partial}\varphi) - \bar{\partial}^2\varphi$$

satisfy the equations

$$\bar{\partial}T = 0 \qquad \partial\bar{T} = 0$$

Hence, the field $T = T(z)$ is chiral and $\bar{T} = \bar{T}(\bar{z})$ is anti-chiral.

$V = e^{-\varphi(z,\bar{z})}$ therefore satisfies the equations

$$\partial^2 V = T(z)\,V \qquad \bar{\partial}^2 V = \bar{T}(\bar{z})\,V$$

From these linear equation one can also find the general solution.

Mechanical Liouville model

The homogeneous Liouville field configurations $\partial_\sigma \varphi = 0$ describe the dynamics of a particle in the exponential potential

$$\ddot{x}(\tau) + 4m^2\, e^{2x(\tau)} = 0$$

The Lagrangian of this model is

$$L = \frac{1}{2}\,\dot{x}^2 - 2m^2\, e^{2x}$$

and the conserved energy becomes

$$E = \frac{1}{2}\,\dot{x}^2 + 2m^2\, e^{2x}$$

The linear system with chiral $T(z)$ and anti-chiral $\bar{T}(\bar{z})$ fields are reduced to the linear equation with a constant coefficient

$$\ddot{V} = 2E\,V$$

Reflection in the Liouville model

The integration of the linear equation provides the solution

$$x(t) = \log \frac{p}{2m \cosh(q + p\tau)}$$

where $p = \sqrt{2E}$ and q is an integration constant.
The variables (p, q) parameterize the space of solutions and since the energy is positive one gets $p > 0$.
The parameter p corresponds to the asymptotic in-momentum of the Liouville particle and $-p$ is the out-momentum, respectively.
The canonical transformation from in to out variables is given by

$$(p_{in}, q_{in}) \;\mapsto\; (p_{out}, q_{out}) = (-p_{in}, -q_{in} + \beta'(p_{in})) \;,$$

and we find the function $\beta(p)$.

Witten's model

This model is given by the equation

$$\partial\bar{\partial}u = u^* \ \frac{\partial u \ \bar{\partial} u}{1 + uu^*}$$

where $u = u_1 + iu_2$ is a complex valued field.
The conformal symmetry of the model:
If $u(z, \bar{z})$ is a solution of the dynamical equation then the field

$$\tilde{u}(z, \bar{z}) = u\left(f(z), \bar{f}(\bar{z})\right)$$

satisfies the same equation.
For homogeneous configurations the field equation reduces to

$$\ddot{u} = \frac{u^* \, \dot{u}^2}{1 + uu^*}$$

and its Lagrangian is

$$L = \frac{\dot{u} \, \dot{u}^*}{1 + uu^*}$$

Witten's model

Conserved quantities are the energy and the angular-momentum

$$E = \frac{\dot{u}\,\dot{u}^*}{1 + uu^*} \qquad\qquad M = \frac{i\,u\,\dot{u}^* - i\,u^*\,\dot{u}}{1 + uu^*}$$

This enables us to rewrite the dynamical equation in a linear form

$$\ddot{u} = 2i\,M\,\dot{u} + 2E\,u$$

Zero energy $E = 0$ corresponds to a rest particle $\dot{u} = 0$. In this case $M = 0$ as well. If $E > 0$, it is easy to check that $2E > M^2$. The integration of the linear system yields

$$u = A\,e^{(iM+P)\tau} + B\,e^{(iM-P)\tau}$$

with $P = \sqrt{2E - M^2} > 0$. One finds $4P^2AB^* = (M + iP)^2$, which implies non-vanishing A and B. Thus, for $E > 0$ particle goes to the boundary of the target space at $\tau \to \pm\infty$ and it rotates around the center $u = 0$, if $M \neq 0$.

Conclusions

We have considered two mechanical systems which correspond to homogeneous fields in Liouville theory and Witten's model.

Symmetries of these field theories provide simple integrability of the corresponding mechanical systems. Namely, the integrability of the systems are reduced to linear equations with constant coefficients.

Liouville model describes only a reflection of a particle, whereas the Witten's model contains both scattering and bound states.

For the Liouville model we have calculated the classical S-matrix.

The aim is to calculate its quantum analog for both models and to find the energy spectrum of the bound states in the Witten's model.